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CAPACITY ALLOCATION IN STATE DEPENDENT QUEUEING NETWORKS

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Abstract— Algorithms are developed to compute optimal space allocation in pedestrian circulation systems modeled as state dependent queueing networks. Series, merge, and split topologies are of interest and various performance measures are computed, such as blocking probabilities at the nodes, throughput, average number of customers in the system, and mean delay. Computational experiments testify to the effectiveness of the algorithm. The results obtained indicate that the pattern of the optimal capacity surprisingly repeats over different topologies while it is also heavily dependent upon the arrival rate. Analytical and simulation results are provided to demonstrate the accuracy of the approach.

Key Words— Optimization problems; algorithms; networks; discrete-event systems; stochastic systems.

1 Introduction

Queueing networks with finite capacity and state dependent services are appropriate tools to model many application problems including those in telecommunications, transportation, manufacturing, and service industries. Often times, finite capacities in the queues and state dependent service rates further increase the complexity of solutions for these systems. In other cases, these assumptions may be relaxed. This paper, however, focuses on applications for which it is fundamental to take into account finite capacities and state dependent services.

In this paper, $M/G/C/C$ state dependent queueing networks are of particular interest, *i.e.*, following Kendall's notation, queues with **M**arkovian arrivals, **G**eneral state dependent services, **C** parallel servers, and the total capacity **C** including the servers. $M/G/C/C$ state dependent queueing network models, see Figure 1, have been used successfully in the past to model vehicular networks (Jain and MacGregor Smith, 1997), pedestrian traffic networks (Mitchell and MacGregor Smith, 2001; Cruz and MacGregor Smith, 2002), and accumulation conveyor systems (Thumsi and MacGregor Smith, 1998).

1.1 Motivation

The use of queueing theory for the analysis of congestion in complex systems has a long and storied existence. In the past, queueing networks have been important tools in the study of traffic light synchronization, in the analysis of vehicles at intersections (Newell, 1965), and in the evaluation of traffic flow by using a simplified deterministic approach (May and Keller, 1967). Nowadays, probably boosted by the increasing speed and reduced costs of modern computer systems, more sophisticated models have been developed, *e.g.*, those models including state dependent services. These models have been used in applications such as pedestrian/vehicular network traffic analysis (Cheah and MacGregor Smith, 1994; Jain and MacGregor Smith, 1997; Cruz and MacGregor Smith, 2002) and synthesis (MacGregor Smith, 1994; MacGregor Smith, 1996; Mitchell and MacGregor Smith, 2001).

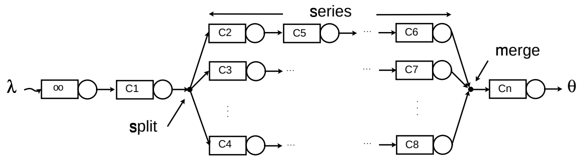

Figure 1. Queueing Network in a Generic Topology

The main reason of this paper is to push forward in the development of algorithms for optimal capacity allocation in $M/G/C/C$ state dependent queueing networks, for a fixed network topology. In particular, the interest lies in pedestrian network applications, configured as series, merge, and split topologies as illustrated in Figure 1. While the focus is on pedestrian networks, extensions to other networks with state dependent service rates should be obvious.

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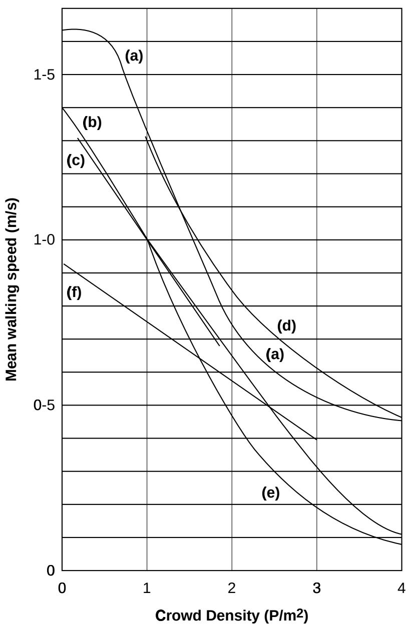


Figure 2. Average Walking Speed

1.2 Outline

Section 2 presents the analytical stochastic model used to describe pedestrian circulation networks. Section 3 gives an overview of the buffer allocation problem and presents the mathematical programming formulation of the service and capacity allocation (SCA) problem. Computational experiments with the algorithm proposed are presented in Section 4. Finally, Section 5 closes the paper with a summary and concluding remarks.

2 Flow Modeling

2.1 Congestion Models

A corridor, a pedestrian way, or a traffic link connecting locations a to b may be considered as a service mechanism for its occupants since it provides the service of moving from point a to point b. The number of servers in parallel, C, equals the nodal capacity which also represents the total number of pedestrians allowed simultaneously in the system, that is:

$$C = \lfloor 5 \times L \times W \rfloor,$$

in which L is the nodal length and W is its width. Notice that 5 ped/m^2 represents the maximum pedestrian density (Tregenza, 1976).

In accordance to Tregenza's empirical studies, the average speed that a pedestrian crosses a traffic link depends on several factors but mainly this speed is a function of the number of occupants therein, as seen in Figure 2. Based on these remarks, linear and exponential congestion models were developed (Yuhaski and MacGregor Smith, 1989) for the average pedestrian walking speed in traffic links, $v(n)$:

$$v(n) = V_1 \frac{C+1-n}{C}, \qquad (1)$$

and

$$v(n) = V_1 \exp\left[-\left(\frac{n-1}{\beta}\right)^{\gamma}\right], \qquad (2)$$

in which

$$\gamma = \ln\left[\frac{\ln(V_a/V_1)}{\ln(V_b/V_1)}\right] / \ln\left(\frac{a-1}{b-1}\right),$$

$$\beta = \frac{a-1}{[\ln(V_1/V_a)]^{1/\gamma}} = \frac{b-1}{[\ln(V_1/V_b)]^{1/\gamma}},$$

V_1 is the average walking speed for lone occupant, assumed 1.5 m/s, V_a is the average walking speed in m/s when crowd density is 2 ped/m^2, $a = 2LW$, V_b is the average walking speed when crowd density is 4 ped/m^2, and $b = 4LW$.

Yuhaski and MacGregor Smith (1989) point out that the exponential model may be adjusted based on 3 points averaged over the 6 curves in Figure 2. Thus, we assumed $V_a = 0.64$ m/s and $V_b = 0.25$ m/s. Other possibilities also exist, *e.g.* non-linear regression or piece-wise linear approximations, but the results would not differ significantly. In fact, Cruz et al. (2001) have asserted the accuracy of these parameters by using a discrete-event digital simulation model. Additionally, it is worthwhile mentioning that Cheah and MacGregor Smith (1994) successfully extend the exponential model to represent bi-directional and multi-directional pedestrian flows by using slightly different values for V_a and V_b.

2.2 Analytical Model for a Corridor

It is appropriate to describe single corridors as an $M/G/C/C$ state dependent model because the corridor is of finite capacity and of a very general service mechanism. The limiting probabilities for the number of pedestrians in an *pure* Markovian $M/M/C/C$ queueing model have been developed before by Yuhaski and MacGregor Smith (1989). Later, Cheah and MacGregor Smith (1994) showed the stochastic equivalence between $M/G/C/C$ and $M/M/C/C$ systems. Thus, the probabilities for the number of pedestrians in a corridor, modeled as an $M/G/C/C$ state dependent system, can be written as follows:

$$p(n) = \left\{\frac{[\lambda E(t^1)]^n}{n! f(n) \cdots f(2) f(1)}\right\} p_0, \qquad (3)$$

for $n = 1, 2, \ldots, C$, in which

$$p_0^{-1} = 1 + \sum_{i=1}^{C} \left\{\frac{[\lambda E(t^1)]^i}{i! f(i) \cdots f(2) f(1)}\right\},$$

λ is the arrival rate in ped/s, $E(t^1)$ is the service time for lone occupant in seconds, and $f(n)$ is the service rate $v(n)/V_1$. From Eq. (3), one can derive all performance measures of interest:

$$\begin{cases} \theta &= \lambda\Big(1 - p(C)\Big), \\ E(q) &= \sum_{n=1}^{C} np(n), \\ E(t) &= E(q)/\theta, \end{cases} \quad (4)$$

in which $p(C)$ is the blocking probability, θ is the throughput in ped/s, $E(q)$ is the expected number of customers in the systems (also known as work-in-process, WIP), and $E(t)$ is the expected service (delay) time in seconds.

2.3 Generalized Expansion Method

For the analysis of a complex topology such as that presented in Figure 1, one might want to use some approximation technique since it seems unlikely that an exact method would be available. The Generalized Expansion Method (GEM) proposed by Kerbache and MacGregor Smith (1987) has been successful in similar problems. The GEM is a combination of repeated trials and node-by-node decomposition approximation methods, with a key characteristic that an artificial holding node is added preceding each finite queue in the network in order to register blocked customer that attempt to enter the finite node when it is at capacity. By adding holding nodes, the queueing network is 'expanded' into an equivalent Jackson network, in which each node can then be decomposed and analyzed separately. Details on how the GEM can be adapted to $M/G/C/C$ state dependent queueing networks will not be given here but may be found in the work of Cruz and MacGregor Smith (2002).

3 Problem Formulation and Algorithm

3.1 Mathematical Programming Formulation

Assume that the topology of the network is known beforehand and is defined as a graph $G(N, A)$, in which N is the finite set of nodes (corridors) and A is the finite set of arcs (connections between pair of nodes). The service and capacity allocation (SCA) problem is concerned with how much capacity must be provided in the nodes so that the blocking probability is below a specific threshold. In other words, the SCA problem is to find the smallest integers $C_i \geq 0$ for which $p_i(\mathbf{C}) \leq \varepsilon_i$, for all $i \in N$. Note that the service rate depends on the capacity vector $\mathbf{C}$, either under the linear model, Eq. (1), or the exponential model, Eq. (2). For simplicity, only the exponential model is used in this paper.

The mathematical programming formulation proposed for the SCA problem is the following:

(SCA):

$$z = \min \sum_{\forall\, i \in N} f_i C_i, \quad (5)$$

s.t.

$$p_i(\mathbf{C}) \leq \varepsilon, \quad \forall\, i \in N, \quad (6)$$

$$C_i \in \{0, 1, \ldots\}, \quad \forall\, i \in N, \quad (7)$$

that minimizes the overall allocation cost $\sum_i f_i C_i$, constrained to provide a minimum blocking probability $p_i(\mathbf{C})$ for all nodes.

In spite of the linearity of its objective function, the SCA problem has inherent complications. From a practical point of view, one serious aspect to deal with is the intractability of the expressions for $p_i(\mathbf{C})$ in closed form for any given topology. In a topology such as the one seen in Figure 1, the blocking probability at the ith node depends on all upstream incoming flows and also on the blocking probabilities of all downstream nodes.

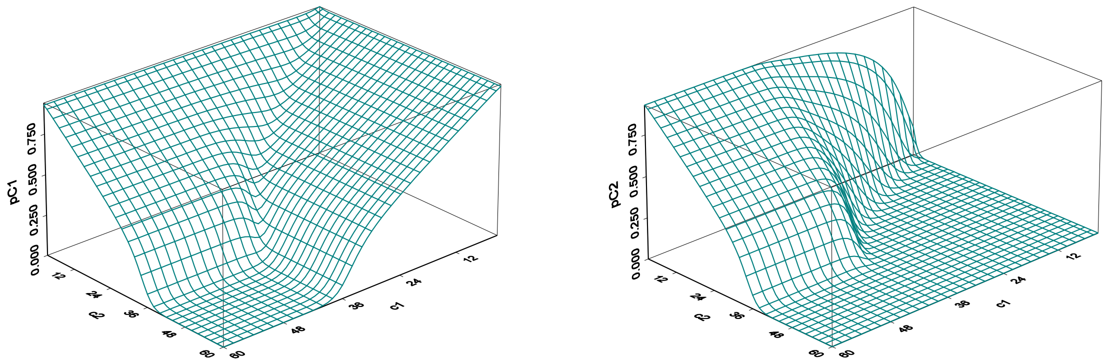
In Figure 3, we can see how complex are the blocking probabilities, as a function of capacities C_1 and C_2, even in a simple 2-node tandem configuration, assumed that the capacity of each node is a function of the width only (remaining unchanged the length). Notice that the 'flat' part of the curves corresponds roughly to the feasible regions, *i.e.*, points for which $p_1(\mathbf{C}) \leq \varepsilon$ and $p_2(\mathbf{C}) \leq \varepsilon$.

3.2 Proposed Algorithm

Many of the approaches already described in the literature on buffer allocation could perhaps be adapted to solve the SCA problem as well. The reader is encouraged to check the new material by MacGregor Smith et al. (2000), for a recent overview on some of the newest advances on this topic. The algorithm proposed here is inspired by those approaches and is shown in Figure 4 in pseudo-code.

Figure 4-a show the main algorithm that implements a variation of the derivative free coordinate search method. All settings are read and an initial feasible solution is found which is to set a large enough capacity to all nodes, in order to make sure that no queue will be blocked at all and that the constraints will be satisfied. For convenience, the initial feasible capacity is in the form 2^M since it will help the local search algorithm, to be described as follows.

The local search is presented in Figure 4-b. First, a recursive labeling step is applied in order to ensure that no node is locally optimized unless all of its predecessor nodes were already optimized. This is necessary because the GEM tends to underestimate the blocking probabilities in situations in which there is a severe bottleneck in the final nodes of the network, as observed by Cruz and MacGregor Smith (2002).



a) node 1 b) node 2

Figure 3. Blocking Probabilities $p_i(\mathbf{C})$ in a 2-Nodes Tandem Configuration

algorithm
  read $G(N, A)$
  read routing probabilities p_{ij}, $\forall\ (i, j) \in A$
  read arrival rates λ_i *and* c_i, $\forall\ i \in N$
    /* find initial *feasible* vector $\mathbf{C}$ */
  for $\forall\ i \in N$ **do**
    $C_i^{\text{opt}} \leftarrow 2^M$
  end for
    /* search optimum solution */
  iter $\leftarrow 0$
  repeat
    iter $\leftarrow$ iter $+ 1$
      /* optimizes i-th queue */
    for $\forall\ i \in N$ **do**
      OptQueue(i,$\mathbf{C}^{\text{sup}}$)
    end for
      /* update best solution */
    if $g(\mathbf{C}^{\text{sup}}) < g(\mathbf{C}^{\text{opt}})$ **then**
      $\mathbf{C}^{\text{opt}} \leftarrow \mathbf{C}^{\text{sup}}$
      unmark all nodes
    else
      exit
    end if
  end repeat
  write $\mathbf{C}^{\text{opt}}$
end algorithm

a) network optimization

algorithm OptQueue(i,$\mathbf{C}^{\text{sup}}$)
    /* recursive labeling step */
  for $\forall\ (j, i) \in A$ **do**
    if *node* j *is unmarked* **then**
      OptQueue(j,$\mathbf{C}^{\text{sup}}$)
      mark node j
    end if
  end for
    /* isolate optimum */
  $j \leftarrow M_i$
  $C_i^{\text{inf}} \leftarrow C_i^{\text{sup}} \leftarrow 2^j$
  while $p_i(\mathbf{C}^{\text{sup}}) \not\leq \varepsilon_i, \forall\ i \in N$
    $j \leftarrow j + 1$
    $C_i^{\text{sup}} \leftarrow 2^j$
  end while
    /* narrow interval */
  $\mathbf{C}^{\text{can}} \leftarrow \mathbf{C}^{\text{sup}}$
  while $(C_i^{\text{sup}} - C_i^{\text{inf}}) > 1$
    $C_i^{\text{can}} \leftarrow (C_i^{\text{inf}} + C_i^{\text{sup}})/2$
    if $p_i(\mathbf{C}^{\text{can}}) \leq \varepsilon_i, \forall\ i \in N$ **then**
      $C_i^{\text{sup}} \leftarrow C_i^{\text{can}}$;
    else
      $C_i^{\text{inf}} \leftarrow C_i^{\text{can}}$;
    end if
  end while
end algorithm

b) single queue optimization

Figure 4. SCA Problem Resolution Algorithm

Then, the single queue optimization algorithm isolates the optimum by coming up with an interval in which the inferior limit capacity C_i^{inf} is infeasible and the superior limit capacity C_i^{sup} is feasible. The next step is to reduce the initial interval up to that point they differ by the unity. Then, the superior limit is the smallest capacity that complains with the blocking probability requirements for the ith queue.

4 Computational Experiments

The proposed algorithms were coded in C++, a flexible and efficient programming language. All computational experiments were carried out on a PC, CPU Pentium II 400 MHz, 64 MB RAM, Windows NT 4.0 operating system. Several topologies were of interest, series, merges, and splits. Experiments were done for 3 and 5 node networks. Arrival rates of 1, 2, and 4 ped/sec were considered. Additionally, the discrete-event digital simulation model developed by Cruz et al. (2001) was run to confirm the accuracy of the solutions generated. All simulations were run for 22,000 seconds, with a *burn-in* period of 2,000 seconds, and 30 replications were performed in order to compute 95% confidence intervals.

For simplicity, we used only the exponential model, Eq. (2), and only networks of nodes of identical lengths. All nodes were assumed 8 meters in length. The widths are the decision variables. A threshold blocking probability of 0.1% (0.001) and a unitary allocation cost f_i were used, for all nodes. For each network considered, simulations were performed for the optimal solution and around the optimal, with slight perturbations on the capacity of one of the queues. The cpu times expended are provided only for the simulations. The algorithms for analysis and optimization spent less than 10 seconds in the worst case.

Table 1 shows the results obtained for tandem (series) topologies. Notice that for series topolo-

Table 1. Tandem (series) Topologies

λ	**C** (ped)	**W** (m)	max $p(\mathbf{C})$ GEM	simulation average	simulation 95% CI	simulation cpu (mm:ss)
1.0	41 → 44 → 45	1.04 → 1.11 → 1.14	0.0024	0.0021	[0.0013 ;0.0029]	0:55
	42 → 45 → 46	1.06 → 1.14 → 1.16	0.0010*	0.0014	[-0.0001;0.0028]	1:47
	43 → 46 → 47	1.09 → 1.16 → 1.19	0.0004	0.0004	[0.0001 ;0.0006]	2:01
	41 → 44 → 45 → 46 → 47	1.04 → 1.11 → 1.14 → 1.16 → 1.19	0.0024	0.0021	[0.0013 ;0.0029]	6:22
	42 → 45 → 46 → 47 → 48	1.06 → 1.14 → 1.16 → 1.19 → 1.21	0.0010*	0.0024	[-0.0003;0.0051]	3:56
	43 → 46 → 47 → 48 → 49	1.09 → 1.16 → 1.19 → 1.21 → 1.24	0.0004	0.0004	[0.0001 ;0.0006]	3:06
2.0	78 → 78 → 81	1.96 → 1.96 → 2.04	0.0022	0.0128	[0.0017 ;0.0240]	5:16
	79 → 79 → 82	1.99 → 1.99 → 2.06	0.0010*	0.0029	[-0.0002;0.0059]	4:59
	80 → 80 → 83	2.01 → 2.01 → 2.09	0.0004	0.0095	[-0.0042;0.0231]	5:01
	78 → 78 → 81 → 82 → 82	1.96 → 1.96 → 2.04 → 2.06 → 2.06	0.0023	0.0128	[0.0017 ;0.0240]	10:19
	79 → 79 → 82 → 83 → 83	1.99 → 1.99 → 2.06 → 2.09 → 2.09	0.0010*	0.0029	[-0.0002;0.0059]	9:36
	80 → 83 → 83 → 84 → 84	2.01 → 2.01 → 2.09 → 2.11 → 2.11	0.0004	0.0095	[-0.0042;0.0231]	9:53
4.0	150 → 151 → 154	3.76 → 3.79 → 3.86	0.0021	0.0000	[0.0000 ;0.0000]	16:49
	151 → 152 → 155	3.79 → 3.81 → 3.89	0.0010*	0.0000	[0.0000 ;0.0000]	17:04
	152 → 153 → 156	3.81 → 3.84 → 3.91	0.0005	0.0000	[0.0000 ;0.0000]	16:46
	150 → 151 → 154 → 156 → 157	3.76 → 3.79 → 3.86 → 3.91 → 3.94	0.0021	0.0000	[0.0000 ;0.0000]	41:42
	151 → 152 → 155 → 157 → 158	3.79 → 3.81 → 3.89 → 3.94 → 3.96	0.0010*	0.0000	[0.0000 ;0.0000]	40:24
	152 → 153 → 156 → 158 → 159	3.81 → 3.84 → 3.91 → 3.96 → 3.99	0.0005	0.0000	[0.0000 ;0.0000]	40:55

* optimization algorithm best solution

gies, nodes tend to be wider at the end of topology. The effect of blocking at end nodes is amplified back at the upstream nodes so that some extra space must be allocated in order to avoid the effect and to meet the performance requested. Additionally, it is remarkable that this progressive increasing allocation pattern repeats over longer networks.

Table 2 shows the results obtained for split and merge topologies. For split topologies, unbalanced splitting probabilities of 0.6 and 0.4 were considered, and, as a consequence, unbalanced allocations were obtained for the nodes following the splitting node. For merges, the arrival rates at the two front nodes were balanced and, as expected, balanced capacity were allocated there. In all 5-node networks, a similar effect as in the series topologies (*i.e.*, a progressively increasing capacity allocation) was observed in the tandem links. Finally, one can surprisingly see a 'economy-of-scale' effect since neither the capacity at the node after merging equals the sum of capacities of nodes just before merging in the merge topologies, nor the sum of capacities of nodes that follow a split equals the capacity of the node before splitting.

5 Concluding Remarks

A methodology based on $M/G/C/C$ state-dependent queueing systems, suitable for analysis and synthesis of systems subject to congestion effects, in particular, pedestrian networks, was presented. The importance of the model was stressed and a short review of recent results on the area was presented. In detail, the application of the model to pedestrian network planning was discussed. Computational results were provided to demonstrate the effectiveness of the approach.

Many research questions remain. The algorithm must be tested under different topologies and blocking probabilities, as well as under heavier and lighter arrival rates. Another possibility is to extend the congestion model to modeling vehicular networks. These are only some possible directions for future research in the area.

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Table 2. Split & Merge Topologies

λ	**C** (ped)	**W** (m)	max $p(\mathbf{C})$ GEM	simulation average	simulation 95% CI	simulation cpu (mm:ss)
1.0	41 ↗29 ↘21	1.04 ↗0.74 ↘0.54	0.0025	0.0018	[0.0011 ; 0.0024]	1:04
	42 ↗30 ↘22	1.06 ↗0.76 ↘0.56	0.0010*	0.0008	[0.0004 ; 0.0012]	1:03
	43 ↗31 ↘23	1.09 ↗0.79 ↘0.59	0.0004	0.0004	[0.0001 ; 0.0006]	1:02
	41 ↗29 → 31 ↘21 → 23	1.04 ↗0.74 → 0.79 ↘0.54 → 0.59	0.0025	0.0018	[0.0011 ; 0.0024]	1:37
	42 ↗30 → 32 ↘22 → 24	1.06 ↗0.76 → 0.81 ↘0.56 → 0.61	0.0010*	0.0008	[0.0004 ; 0.0012]	1:36
	43 ↗31 → 33 ↘23 → 25	1.09 ↗0.79 → 0.84 ↘0.59 → 0.64	0.0004	0.0004	[0.0001 ; 0.0006]	1:35
2.0	78 ↗49 ↘35	1.96 ↗1.24 ↘0.89	0.0020	0.0046	[0.0028 ; 0.0065]	2:58
	79 ↗50 ↘36	1.99 ↗1.26 ↘0.91	0.0009*	0.0031	[0.0010 ; 0.0051]	2:54
	80 ↗51 ↘37	2.01 ↗1.29 ↘0.94	0.0004	0.0007	[-0.0001; 0.0015]	2:50
	78 ↗49 → 50 ↘35 → 37	1.96 ↗1.24 → 1.26 ↘0.89 → 0.94	0.0023	0.0107	[0.0060 ; 0.0154]	4:50
	79 ↗50 → 51 ↘36 → 38	1.99 ↗1.26 → 1.29 ↘0.91 → 0.96	0.0010*	0.0037	[0.0012 ; 0.0063]	4:38
	80 ↗51 → 52 ↘37 → 39	2.01 ↗1.29 → 1.31 ↘0.94 → 0.99	0.0004	0.0009	[-0.0004; 0.0021]	4:31
1.0	23 ↘, 23 ↗ → 43	0.59 ↘, 0.59 ↗ → 1.09	0.0023	0.1067	[0.0401 ; 0.1733]	1:39
	24 ↘, 24 ↗ → 44	0.61 ↘, 0.61 ↗ → 1.11	0.0008*	0.0481	[-0.0019; 0.0981]	1:21
	25 ↘, 25 ↗ → 45	0.64 ↘, 0.64 ↗ → 1.14	0.0003	0.0289	[-0.0109; 0.0687]	1:15
	23 → 24 ↘, 23 → 24 ↗ → 44	0.59 → 0.61 ↘, 0.59 → 0.61 ↗ → 1.11	0.0028	0.0342	[0.0037 ; 0.0647]	2:05
	24 → 25 ↘, 24 → 25 ↗ → 45	0.61 → 0.64 ↘, 0.61 → 0.64 ↗ → 1.14	0.0010*	0.0394	[-0.0028; 0.0816]	2:20
	25 → 26 ↘, 25 → 26 ↗ → 46	0.64 → 0.66 ↘, 0.64 → 0.66 ↗ → 1.16	0.0003	0.0042	[-0.0006; 0.0091]	1:39
2.0	41 ↘, 41 ↗ → 80	1.04 ↘, 1.04 ↗ → 2.01	0.0024	0.0284	[-0.0108; 0.0676]	3:25
	42 ↘, 42 ↗ → 81	1.06 ↘, 1.06 ↗ → 2.04	0.0010*	0.0093	[-0.0089; 0.0275]	3:04
	43 ↘, 43 ↗ → 82	1.09 ↘, 1.09 ↗ → 2.06	0.0004	0.0005	[0.0002 ; 0.0007]	2:51
	41 → 44 ↘, 41 → 44 ↗ → 82	1.04 → 1.11 ↘, 1.04 → 1.11 ↗ → 2.06	0.0024	0.0088	[-0.0084; 0.0259]	5:23
	42 → 45 ↘, 42 → 45 ↗ → 83	1.06 → 1.14 ↘, 1.06 → 1.14 ↗ → 2.09	0.0010*	0.0009	[0.0005 ; 0.0013]	4:32
	43 → 46 ↘, 43 → 46 ↗ → 84	1.09 → 1.16 ↘, 1.09 → 1.16 ↗ → 2.11	0.0004	0.0005	[0.0002 ; 0.0007]	4:27

* optimization algorithm best solution

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