PERFORMANCE EVALUATION OF $GI^X/M/c/N$ SYSTEMS THROUGHT KERNEL ESTIMATION

G. M. Gontijo, G. S. Atuncar, F. R. B. Cruz
Departamento de Estatística – Universidade Federal de Minas Gerais
Av. Antônio Carlos, 6627, 31.270-901 – Belo Horizonte - MG
gmgontijo@ufmg.br, gregorio@est.ufmg.br, fcruz@ufmg.br

ABSTRACT

This paper extends the analysis of queuing systems for real situations, where no one knows the pattern of arrivals of customers. Thus, for real systems, one must understand how the choice of a method of estimation influences the configuration of the system. In this manner, we evaluate some algorithms to estimate through kernel smoothing some performance measures of a $GI^X/M/c/N$ system, such as the invariant distribution of probability of the number of customers in the system, the blocking probability, the average queue size, and the average client queue time. Thus, we hope to adequately plan queuing systems and improve their performance.

KEYWORDS. Statistics, kernel estimator, performance evaluation.

1. Introduction

1.1. Preliminaries

When we are managing real queuing systems, generally we don't know *a priori* the behavior of arrival process and service process. Mathematical modeling depends of this information and, naturally, there are several methods to obtain them. The most widely used methods are those that attempt to explain the density functions of interarrival time and service time through parametric statistical models.

Nevertheless, real data rarely fit straightly into a parametric model, or when fitted, makes a very complex mathematical modeling. Kalashnikov (1994) warned that "many parts of the theory of queues were developed as a 'pure science' with no practical application." Bareche & Aïssani (2008) assert that real systems are "generally very complicated, so their analysis cannot lead to analytical results or it leads to complicated results which are not useful in practice ".

An alternative approach is nonparametric. Some nonparametric methods that have taken great interest are those that use kernel smoothing. Kernel estimators provide a simple way of finding structure in data sets without the imposition of a parametric model (Wand & Jones, 1995), which gives us flexibility to handle any data structure. There is some extensive literature about queuing systems or discussing kernel smoothing, but virtually there is no literature with the two concepts together. We will mention some of the most relevant papers.

In Kendall (1951) notation, the focus of this paper is $GI^X/M/c/N$ queues. In this situation, the interarrival times are independent and do not follow any specific distribution. The service times follow a Markov process (exponential time), we have c identical servers working in parallel and a maximum capacity N of the system. X is a random variable representing the size of the group arrival. Takács (1962) analyzed a closed solution for various systems that have non-specific distributions, including some multiserver queues such as $GI/M/c$ and $M/G/c$; Hokstad (1975) established some closed form results to the $GI/M/c/N$ system; Chaudhry & Templeton (1983) analyzed various types of queue with bulk arrivals; Vijaya Laxmi & Gupta (2000) defined the linear equations needed to solve the $GI^X/M/c/N$ system; Zhao (2004) proposed a closed form solution for the $GI^X/M/c$ system; Bareche & Aïsani (2008) proposed a method to evaluate the proximity of GI/M/1 and M/M/1 systems when the density of the interarrival time is estimated by kernel estimators.

The reader can find more information on kernel smoothing in Wand & Jones (1995). Zhang, Karunamuni & Jones (1999) proposed a boundary corrected kernel estimator based on pseudodata generation, transformation and reflection around the Y axis; Chen (2000) proposed the use of gamma kernel in order to avoid the boundary problem; Scaillet (2004) studied the application of other asymmetric kernels; Bouezmarni & Scaillet (2005) were concerned about the consistency of these asymmetrical estimators.

Consequently, there is a huge interest in investigating the behavior of queuing systems, especially those of type $GI^X/M/c/N$, where the arrival process and service process are evaluated through kernel smoothing. Queuing systems like those could be used in situations where we have relative control over how the servers work, but we don't know primary how customers arrive to the system.

1.2. Key Contributions

Among the contributions of this paper, we could mention the following:

- Presentation of an updated bibliography about kernel-estimators and queuing systems;
- Development of algorithms to calculate performance measures of queuing systems, where the density of the interarrival time is estimated by kernel-estimators;

- Evaluation of the performance of these algorithms as function of the kernel-estimator, the smoothing window, the intensity rate, and the sample size.

1.3. Structure of Paper

The organization is presented below. In Section 2 we present the fundamental concepts relevant to understand the model of the queuing system proposed. The concept of estimation by kernel-estimators is also discussed in this section along with some models adopted and the problem of choosing the smoothing parameter. The steps developed for the calculation of estimates of system performance are presented in Section 3 together with the description and the results of comparative simulations with the different methods discussed. Finally, in Section 4 we present the main conclusions with some ideas for future work in the area.

2. Fundamental Concepts

2.1. The $GI^X/M/c/N$ model with partial blocks

Vijaya Laxmi & Gupta (2000) described a generalization of the system *GI/M/c* when customers arrive in groups of size X with $P(X = i) = g_i$ $(i \geq 1)$ and mean $E(X) = \bar{g}$. Let B_n be the number of clients who were served between the arrival of the nth customer and its successor. Therefore, the number of clients the nth customer finds in the system at the arrival, Y_n, would depend on X_n and B_n, such that $Y_{n+1} = [Y_n + X_n - B_n]^+$. Since Y_{n+1} depends only of Y_n, B_n and X_n, not Y_{n-1}, Y_{n-2}, etc., the stochastic process $\{Y_n\}$ is a Markov Chain.

The $GI^X/M/c/N$ is a finite capacity system such that a customer that arrives to the saturated system is refused with a probability that we will call P_{BL}. Partial blocks means the case which an arrival group with size greater than the remaining spots of the system is denied partially according to the number of these remaining vacancies, thus completing the system.

Vijaya Laxmi & Gupta (2000) report that when the traffic intensity rate $\rho = (\lambda \bar{g} / c\mu)$ is smaller than 1, this Markov Chain has an invariant distribution of probability $\pi_k = \lim_{n \to \infty} P(Y_n = k)$ $(k = 0,1,2,\ldots)$ associated with the number of clients an arbitrary customer finds the system at the arrival. The π_k 's are often called prearrival probabilities.

The prearrival probabilities may be determined by a system of linear equations

$$\pi_k = \sum_{j=0}^{N} p_{jk} \pi_j \quad (k = 0,1,2,\ldots,N-1) \quad \text{and} \quad \sum_{j=0}^{N} \pi_j = 1$$

$$\begin{bmatrix} (p_{0,0}-1) & p_{1,0} & \cdots & p_{N,0} \\ p_{0,1} & (p_{1,1}-1) & & \vdots \\ \vdots & & \ddots & \vdots \\ p_{0,N-1} & \cdots & \cdots & (p_{N,N-1}-1) \\ 1 & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \vdots \\ \vdots \\ \pi_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

where p_{jk} are named transition probabilities such that

$$p_{jk} = P\{Y_{n+1} = k \mid Y_n = j\} = \sum_{i=1}^{\infty} P\{X_n = i, Y_{n+1} = k \mid Y_n = j\} = \sum_{i=1}^{\infty} g_i P\{Y_{n+1} = k \mid Y_n = j, X_n = i\}$$

$$= \sum_{i=1}^{\infty} g_i P(B_n = j-k+i) = \sum_{i=1}^{\infty} g_i \int_0^{\infty} \pi_{j+i,k}(z) d\tau(z)$$

$i \geq k-j$, $j \leq N$, $k \leq N$ and $\pi_{j+i,k}(z)$ is the serving probability of $(j-k+i)$ clients under the assumption of the interarrival time $\tau_n = z$. We shall analyze how $\pi_{j+i,k}(z)$ behave.

When $j+i \geq c$ and $k \geq c$, there will be more clients than the servers can handle at the whole interval. Since the service process is Markovian, we can then take server group as a single unit who serves customers at a rate $c\mu z$ and this transition probability by a Poisson distribution:

$$\pi_{j+i,k}(z) = e^{-c\mu z} \frac{(c\mu z)^{j+i-k}}{(j+i-k)!}$$

When $j+i < c$ all clients within the system are being serviced and only k customers will remain in the system to time. Knowing that the probability of a service time greater than z is $e^{-\mu z}$, we can describe this transition probability as Binomial distribution:

$$\pi_{j+i,k}(z) = \binom{j+i}{k} e^{-k\mu z} \left(1-e^{-\mu z}\right)^{j+i-k}$$

And when $j+i \geq c$ and $k < c$, there will be $(j+i-c)$ customers waiting and c customers being served at the beginning of interval, but $(c-k)$ spots at the end. Let y be the interval ended immediately before $(j+i-c+1)$ clients being served. If each service time is exponentially distributed with rate $c\mu$, then y is distributed as a gamma of shape $(j+i-c+1)$ and rate $c\mu$. The other c customers will be served in a time $(z-y)$ and only k will remain. The transition probability of this subinterval will follow a Binomial with success probability $e^{-\mu(z-y)}$. $\pi_{j+i,k}(z)$ can be obtained by the convolution of these two variables:

$$\pi_{j+i,k}(z) = \int_0^z e^{-c\mu y} \frac{(c\mu)^{j+i-c+1} y^{j+i-c}}{(j+i-c)!} \binom{c}{k} e^{-k\mu(z-y)} (1-e^{-\mu(z-y)})^{c-k} dy$$

$$= \pi_{j+i,k}(z) = \binom{c}{k} e^{-k\mu z} \int_0^z \frac{(c\mu y)^{j+i-c}}{(j+i-c)!} e^{k\mu y} e^{-c\mu y} (1-e^{\mu y} e^{-\mu z})^{c-k} c\mu dy$$

$$= \pi_{j+i,k}(z) = \binom{c}{k} e^{-k\mu z} \int_0^z \frac{(c\mu y)^{j+i-c}}{(j+i-c)!} \left[e^{-\mu y} (1-e^{\mu y} e^{-\mu z})\right]^{c-k} c\mu dy$$

$$= \pi_{j+i,k}(z) = \binom{c}{k} e^{-k\mu z} \left[\int_0^z \frac{(c\mu y)^{j+i-c}}{(j+i-c)!} \left(e^{-\mu y} - e^{-\mu z}\right)^{c-k} c\mu dy\right].$$

Therefore, we can get Vijaya Laxmi & Gupta (2000) transitions probabilities when we take each transition probability $\pi_{j+i,k}(z)$ as $\pi_{N,k}(z)$ while $i > N-j$.

$$p_{jk} = \begin{cases} \sum_{i=\max\{1,k-j\}}^{N-j} \beta_{j+i-k} g_i + \beta_{N-k} \sum_{k=N-j+1}^{\infty} g_i \,, & k \geq c \\ \sum_{i=\max\{1,k-j\}}^{N-j} V_{j+i,k} g_i + V_{N,k} \sum_{k=N-j+1}^{\infty} g_i \,, & 0 < k < c \\ 1 - \sum_{r=1}^{N} p_{jr} \,, & k = 0 \end{cases}$$

They named the integration of $\pi_{j+i,k}(z)$ with β_{j+i-k} and $V_{j+i,k}$:

$$V_{j,k} = \begin{cases} 0\,, & j < k < c \\ \int_0^{\infty} \binom{j}{k} e^{-k\mu z} \left(1 - e^{-\mu z}\right)^{j-k} dA(z)\,, & k \leq j \leq c \\ \int_0^{\infty}\int_0^{z} \binom{c}{k} e^{-k\mu z} \frac{(c\mu y)^{j-c}}{(j-c)!} c\mu \left(e^{-\mu y} - e^{-\mu z}\right)^{c-k} dy dA(z)\,, & k < c < j \end{cases}$$

$$\beta_r = \int_0^{\infty} e^{-c\mu z} \frac{(c\mu z)^r}{r!} dA(z), \quad r \geq 0$$

2.2. Kernel-estimators

Suppose we have a sample of the interarrival times X_1 ,..., X_n that have density $\tau(t)$ unknown. The kernel-estimator is an analytical tool that provides a very effective way of revealing the structure behind the sample.

2.2.1. The gamma kernel-estimator

Chen (2000) recently suggested an asymmetric kernel with naturally varying shape, in a way to never allocate weight for negative values. The gamma kernel-estimators, in particular, are always non-negative, free of boundary bias and achieve the optimal rate of convergence for the MSE in the class of non-negative kernel-estimators. Bouezmarni & Scaillet (2005) showed that the estimator is consistent and able to avoid boundary bias. Be $K_G(p,q)$ the gamma density function with shape p and rate q. The gamma kernel considered is

$$K_G\left(\frac{t}{b}+1, b\right)(X_j) = \frac{X_j^{t/b} e^{-X_j/b}}{b^{(t/b)+1}\Gamma\left[(t/b)+1\right]}$$

where b is a smoothing parameter satisfying the condition that $b \to 0$, $nb \to \infty$ as $n \to \infty$. The gamma kernel estimator is

$$\hat{\tau}(t;b) = n^{-1} \sum_{j=1}^{n} K_G\left(\frac{t}{b}+1, b\right)(X_j).$$

The smoothing parameter b is critical for the overall performance of the kernel-estimator considered. A smaller b leads to a relatively jagged density, while a larger one results in a smoother looking. There are several methods to determine the best fit and they start from a minimization of the mean integrated squared error (MISE) of $\hat{\tau}(t;b)$ or its asymptotic behavior (AMISE).

The Least Squares Cross Validation (LSCV) method starts from the MISE expansion

$$MISE\{\hat{\tau}(x;b)\} = E\int \hat{\tau}(x;b)^2\,dx - 2E\int \hat{\tau}(x;b)\tau(x)dx + E\int \tau(x)^2\,dx.$$

The minimization of the first term is equivalent to the minimization of

$$MISE\{\hat{\tau}(x;h)\} - E\int \tau(x)^2\,dx = E\left[\int \hat{\tau}(x;h)^2\,dx - 2\int \hat{\tau}(x;h)\tau(x)dx\right].$$

The right-hand side is unkown since it depends on τ. However, it can be shown that an unbiased estimator for this quantity is

$$LSCV(h) = \int \hat{\tau}(x;h)^2\,dx - 2n^{-1}\sum_{i=1}^{n}\hat{\tau}_{-i}(X_i;h)$$

where $\hat{\tau}_{-i} = (X_i;h)$ is the density estimate based on the sample with X_i deleted, often called the “leave-one-out” density estimator. However, this estimator suffers from high variation.

An alternative parameter selector will consider the asymptotic behavior of the MISE of the gamma kernel estimator. Chen (2000) uses some aspects of the gamma distribution and Taylor expansion to determine the MISE in this case. It is thus defined as

$$MISE(\hat{\tau}) = b^2\int_0^\infty \left\{x\tau'(x) + \frac{1}{2}x\tau''(x)\right\}^2 dx + (2n\sqrt{b\pi})^{-1}\int_0^\infty x^{-1/2}\tau(x)dx + o(n^{-1}b^{-1/2} + b^2)$$

The asymptotic MISE disregards the last term. So the optimal b which minimizes the leading terms above is

$$b_{AMISE} = \left[\frac{\left(2\sqrt{\pi}\right)^{-1}\int_0^\infty x^{-1/2}\tau(x)dx}{4n\int_0^\infty \left(x\tau'(x) + 2^{-1}x\tau''(x)\right)^2 dx}\right]^{2/5}$$

where the functions τ, τ' e τ'', are unknown. These quantities have been computed from the fitted gamma density with parameters adjusted from the sample. This solution still requires further studies, but our paper has already shown some performance results.

2.2.2. Zhang, Karunamuni & Jones (1999) Estimator

Zhang, Karunamuni & Jones (1999) submited a model that works particularly well when $\tau(0) > 0$ and combine pseudo data creation, transformation and reflection around Y axis, consisting of three steps:

• Transform the original data X_1 ,..., X_n to $g(X_1)$,..., $g(X_n)$ while keeping the original data where g is a nonnegative, continuous and monotonically increasing function from $[0,\infty)$ to $[0,\infty)$. Based on extensive simulations, the transformation that is best suited to varied types of densities was $g(x) = x + dx^2 + Adx^3$, where $A > 1/3$ and $d = f'(0)/f(0)$.

• Reflection of pseudo data $g(X_1)$,..., $g(X_n)$ around the origin.

• Based on the enlarged data sample $-g(X_1)$,..., $-g(X_n)$, X_1 ,..., X_n the new estimator is defined as

$$\hat{\tau}_n(x,h) = \frac{1}{nh}\sum_{j=1}^{n}\left\{K\left(\frac{x - X_j}{h}\right) + K\left(\frac{x + g(X_j)}{h}\right)\right\} \qquad x \ge 0$$

where h is an smoothing parameter, K is a symetric probability function with support [-1, 1] as Epanechnikov kernel

$$K(t) = \frac{3}{4}(1 - t^2)I_{[-1,1]}.$$

Notice that the transformation g defined above is not available in practice, because d is unknown. The a good estimator can be obtained when d is writen as $(d/dx)\log f(x)|_{x=0}$. So

$$d_n = \frac{\log f_n(h) - \log f_n(0)}{h},$$

where

$$f_n(h) = f_n^*(h) + \frac{1}{n^2}, \quad f_n(0) = \max\left(f_n^*(0), \frac{1}{n^2}\right),$$

$$f_n^*(h) = \frac{1}{nh}\sum_{j=1}^{n} K\left(\frac{x - X_j}{h}\right), \quad f_n^*(0) = \frac{1}{nh_0}\sum_{j=1}^{n} K_0\left(\frac{X_j}{h_0}\right),$$

and K_0 is a so-called endpoint kernel satisfying

$$\int_{-1}^{0} K_0(t)dt = 1, \quad \int_{-1}^{0} tK_0(t)dt = 0, \quad \int_{-1}^{0} t^2 K_0(t)dt \ne 0$$

and

$$h_0 = \left\{\frac{\left[\int_{-1}^{1} K(t)dt\right]^2 \int_{-1}^{0} K_0(t)^2\,dt}{\left[\int_{-1}^{0} t^2 K_0(t)dt\right]^2 \int_{-1}^{1} K(t)^2\,dt}\right\} h.$$

Zhang, Karunamuni & Jones (1999) asserted that for $t \ge h$, the effect of reflected pseudo data is insignificant and the estimator can be reduced to the Parzen-Rosenblatt estimator.

$$\hat{\tau}_n(t,h) = \frac{1}{nh}\sum_{j=1}^{n} K\left(\frac{t - X_j}{h}\right)$$

They also stated that

$$\int_0^\infty \hat{\tau}_n(t)dt = 1 + \frac{1}{n}\sum_{i=1}^{n}\int_{-X_i/h}^{-g(X_i)/h} K(z)dz\,.$$

Thus, $\hat{\tau}_n(t)$ only integrates to 1 when d_n = 0, so $g_n(X_i) = X_i$, or when $X_i = 0$ for all X_i's, because $g_n(0) = 0$. However, when $n \to \infty$, both limits of the second term will eventually converge to 0 and $\hat{\tau}_n(t)$ will integrates to 1 asymptotically.

Zhang & Karunamuni (1998) used the endpoint kernel

$$K_0(t) = 12(1+t)\left(\frac{1}{2}+t\right)I_{[-1,0]}$$

and showed that this kernel minimizes the MSE when estimating $\tau(0)$. Therefore, $h_0 = 2h$ is approximately the optimal smoothing parameter for estimating $\tau(0)$ except when $\tau(0) = 0$.

Chiu (1991) came up with a parameter selector method that considered the optimal h which minimizes the asymptotic MISE when K is a symmetric probability function with up to the fourth moment finite

$$h_{AMISE} = \left[\frac{\int K(x)^2 dx}{n\left(\int x^2 K(x)dx\right)^2 \int \tau''(x)^2 dx}\right]^{1/5}$$

where the functional $\int \tau''(x)^2 dx$ is unknown.

His "plug in" method consists in estimate this quantity through the characteristic function of the sample

$$\hat{\varphi}(\lambda) = n^{-1}\sum_{j=1}^{n} e^{i\lambda X_j}$$

and calculate the optimal h through the formula above . The characteristic function of τ is

$$\varphi(\lambda) = \int e^{i\lambda x}\tau(x)dx\,.$$

By the Inversion Formula we have

$$\tau(x) = (2\pi)^{-1}\int e^{i\lambda x}\varphi(\lambda)d\lambda$$

thus,

$$\tau''(x) = (2\pi)^{-1} \int \lambda^2 e^{i\lambda x} \varphi(\lambda) d\lambda \,,$$

and,

$$\int \tau''(x)^2 dx = \int [(2\pi)^{-1} \int \lambda^2 e^{i\lambda x} \varphi(\lambda) d\lambda]^2 dx = \int \lambda^4 [(2\pi)^{-1} \int e^{i\lambda x} \varphi(\lambda) d\lambda]^2 dx = \int \lambda^4 [\tau(x)]^2 dx$$

Using Parseval's Identity, we can show that

$$\int \lambda^4 [\tau(x)]^2 dx = (2\pi)^{-1} \int \lambda^4 |\varphi(\lambda)|^2 d\lambda \,.$$

Chiu (1991) introduced a cutoff value Λ for λ, such that $|\hat{\varphi}(\lambda)|^2 < c/n$. Bessegato, Atuncar & Duczmal (2002) affirm that c = 3 minimizes the estimator variance. The final "plug in" estimator is

$$\hat{h}_{AMISE} = \left[\frac{\int K(x)^2 dx}{n\left(\int x^2 K(x) dx\right)^2 \pi^{-1} \int_0^{\Lambda} \lambda^4 \left[|\hat{\varphi}(\lambda)|^2 - n^{-1} \right] d\lambda} \right]^{1/5} .$$

3. Experimental Results

This section presents some results of simulations in R 2.8.0 environment on $GI^X/M/c/N$ systems with partial blocks, where the interarrival time is estimated through the following methods:

- Zhang, Karunamuni & Jones (1999) estimator with "plug in" method;
- Gamma kernel-estimator with LSCV method;
- Gamma kennel-estimator with optimal b_{AMISE}.

To evaluate the performance of the estimators we will compare the Mean Squared Error (MSE) of each pre-arrival estimated probability based on 100 samples of interarrival time. To perform this work we elaborated an algorithm based on the general following steps:

(1) Generate a sample of size n of general interrarrival distribution τ.
(2) Calculate the mean service rate $\mu = \bar{g}\ (\rho c \mathrm{E}(\tau))^{-1}$.
(3) Estimate the optimal smoothing parameter h or b.
(4) Use the kernel density method to estimate the theoretical density function $\tau(x)$.
(5) Find each estimated transition probability.
(6) Solve the linear system

$$\begin{bmatrix} (\hat{p}_{0,0} - 1) & \hat{p}_{1,0} & \cdots & \hat{p}_{N,0} \\ \hat{p}_{0,1} & (\hat{p}_{1,1} - 1) & & \vdots \\ \vdots & & \ddots & \vdots \\ \hat{p}_{0,N-1} & \cdots & \cdots & (\hat{p}_{N,N-1} - 1) \\ 1 & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} \hat{\pi}_0 \\ \hat{\pi}_1 \\ \vdots \\ \vdots \\ \hat{\pi}_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} .$$

The algorithm above was coded in R and is available upon request from the authors for educational and research purposes. The theoretical interrarrival distributions considered in this experiment were:

- Weibull distribution with shape = 2 and rate = 20;
- Gama distribution with shape = 10 ans rate = 2;
- Gamma mixture: 0,45 weight on $\Gamma(5,2)$ and 0,55 weight on $\Gamma(30,1)$.

The first density has $\tau'(0) \neq 0$, the second has $\tau'(0) = 0$ and the last one is bimodal. We choose the weibull and gamma distribution based on their flexibilities. The group size considered is constant (=1) and the system size N = 15. The simulations compared systems with 5 and 10 servers. Samples of size 100 were used. Figures 1 and 2 show some results.

The parameter selector LSCV seems to have a slight advantage under parameter selector b_{AMISE} on the first and last distributions. Zhang, Karunamuni & Jones (1999) estimator had the worst performance on the first and revealed a singular distribution behavior on the last distribution. On the second distribution, there was no difference.

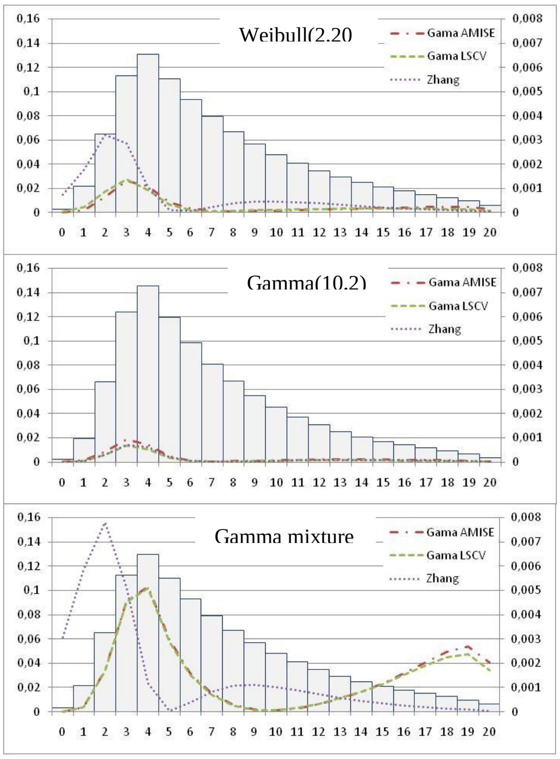

Figure 1. π_n e EQM($\hat{\pi}_n$) with c = 5

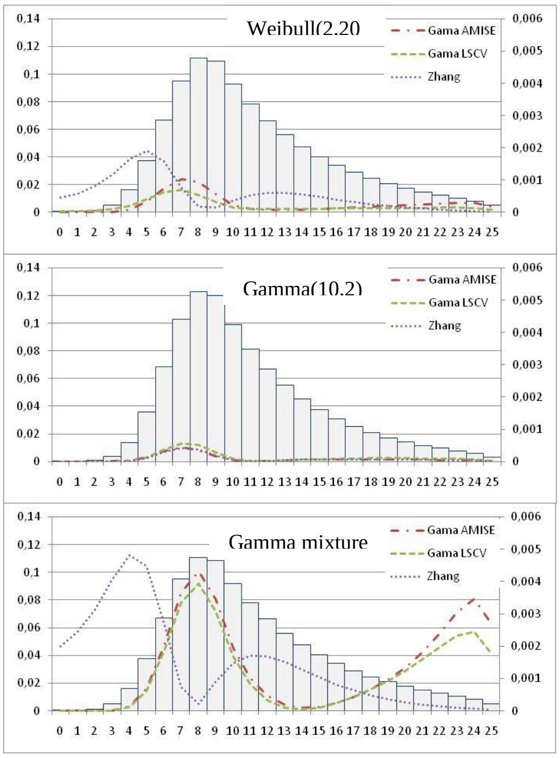

Figure 2. π_n e EQM($\hat{\pi}_n$) with c = 10

4. Concluding Remarks

We studied the adequacy of methods of core estimators to calculate the invariant probability distribution and performance measures of queuing systems that have general distribution of interarrival times. Simulations showed that when $\tau'(0) \neq 0$, the gamma kernel method had better performance. This results point out the understanting that Zhang, Karunamuni & Jones (1999) method does not work well when $\tau(0) > 0$. At the same time, its behavior with the bimodal density showed a very low EQM for probabilities near the maximum state. This might imply a good estimation of the blocking probability and other performance measures.

The method of selection of smoothing parameter on the gama kernel-estimator did not have great impact. A better selector could be developed in a way that the functional of its AMISE optimal parameter would be estimated just like the “plug in” method for symmetric kernels.

Future research can take other possible directions, like the mathematical explanation of how the estimate of the pre-arrival invariant distribution moves away from its real value. An attempt would be from the variance of the bias and variance of each probability estimated. For example, the variance of an estimate of transition probability p_{ij} could depend on the variance of the term $\hat{\beta}_0$, so we would need to find

$$\mathrm{var}(\hat{\beta}_0) = \mathrm{var}\left(\int_0^{\infty} e^{-c\mu z}\, \hat{\tau}(z,h)\,dz\right).$$

Finally, we can take this research to some areas of great practical interest like telephony, health and industry. For example, a medical emergency hall has critical need of resource allocation.

References

Allen, A. O., *Probability Statistics, and Queuing Theory with Computer Science Applications*, Academic Press, Sand Diego, 1990.

Bareche, A. e Aïssani, D. (2008), “Kernel Density in the Study of Strong Stability of the M/M/1 Queuing System”, *Operation Research Letters*, 36, 535-538.

Bessegato, L. F., Atuncar, G. S. e Duczmal, L. H. (2006), *Rotinas em R para Técnicas de Suavização por Núcleo-estimadores*. RTP-01/2006, UFMG/Departamento de Estatística, (http://www.est.ufmg.br/portal/arquivos/rts/rtp0601.pdf)

Bouezmarni, T. e Scaillet, O. (2005), “Consistency of Asymmetric Kernel Density Estimators and Smoothed Histograms with Application to Income Data” *Econometric Theory*, 21, 390–412.

Chaudhry, M. L. e Templeton, J. G. C., *A First Course in Bulk Queues*. Wiley, (1983).

Chen, S. X. (2000), “Probability Density Function Estimation Using Gamma Kernels” *Annals of the Institute of Statistical Mathematics*, 52, 471-480.

Chiu, S. T. (1991), “Bandwidth Selection for Kernel Density Estimation” *Annals of Statistics*, 33, 1883-1905.

Hokstad, P. (1975), “G/M/m queue with finite waiting room” *Journal of Applied Probability*. 12, 779–792.

Kalashnikov, V. V., Mathematical *Methods in Queuing Theory*, Springer, (1994).

Kendall, D. G. (1951), “Some problems in the theory of queues” *Journal of the Royal Statistical Society Serie B*, 13, 151-173

Scaillet, O. (2004), “Density estimation using inverse and reciprocal inverse gaussian kernels” *Journal of Nonparametric* Statistics, 16, 217–226.

Takács, L., *Introduction to the theory of the* queues. Oxford University Press, 1962.

Vijaya Laxmi, P. e Gupta, U.C. (2000), “Analysis of finite-buffer multi-server queues with group arrivals: $GI^X/M/c/N$” *Queueing Systems*, 36, 125-140.

Wand, M. P. e Jones, M. C., *Kernel Smoothing*. CRC Press, 1995.

Zhang, S., Karunamuni, R. J. e Jones, M. C. (1999), “An Improved Estimator of Density Function at the Boundary” *Journal of the American Statistical Association*, 94, 1231-1241.

Zhang, S. e Karunamuni, R. J. (1998), “On Kernel Density Estimation Near Endpoints”, *Journal of Statistical Planning and Inference*, 70, 301-316.

Zhao, Y. (1994), “Analysis of the $GI^X/M/c$ model” *Queueing Systems*, 15, 347-364.