The Beta Log-normal Distribution

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Abstract

For the first time, we introduce the beta log-normal distribution for which the log-normal distribution is a special case. Various properties of the new distribution are discussed. Expansions for the cumulative distribution and density functions that do not involve complicated functions are derived. We obtain expressions for its moments and for the moments of order statistics. The estimation of parameters is approached by the method of maximum likelihood and the expected information matrix is derived. The new model is quite flexible in analyzing positive data as an important alternative to the gamma, Weibull, generalized exponential, beta exponential and Birnbaum-Saunders distributions. The flexibility of the new distribution is illustrated in an application to a real data set.

Keywords: Beta log-normal distribution; Bonferroni curve; Entropy; Log-normal distribution; Lorentz curve; Maximum likelihood estimation; Moment; Observed information matrix; Order statistic.

1 Introduction

Skewed distributions are particulary common when mean values are low, variances large, and values cannot be negative, as is the case, for example, with

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species abundance, lengths of latent periods of infectious diseases, and distribution of mineral resources in the Earth's crust. Such skewed distributions often closely fit the log-normal (LN) distribution (Aitchison and Brown 1957; Crow and Shimizu, 1988; Lee, 1992; Johnson et al., 1994; Sachs, 1997). The LN distribution has interesting applications in many different fields such as agriculture, entomology, atmospheric science, literature, business and reliability, among others. The LN density function is always unimodal and it has the inverted bathtub type hazard function. It is important as a distribution specially when the tail probabilities are of interest in lifetime data analysis. The Weibull and LN distributions are assumed most often in analyzing lifetime data, and in many cases, they are competing with each other. Kim and Yum (2008) used the maximized likelihood and scale invariant procedures to compare and select between these distributions. In addition, various extended Weibull and LN distributions have recently appeared in the literature. See, for example, Flynn (2004), Al-Saleh and Agarwal (2006), Chen (2006), Pham and Lai (2007) and Vera and Díaz-García (2008), among others. They generally fit the data better than two-parameter distributions, although the difference in fits to the data could be insignificant (Algam et al., 2002) or may depend on the selection criterion adopted (Lu et al., 2002). Following this fact, we introduce a new distribution with four parameters, refereed to as the beta log-normal (BLN) distribution, with the hope it will attract wider application in reliability, engineering and other areas of research. This generalization contains as a special sub-model the LN distribution and works as a competitive model to all generalized Weibull distributions.

We shall use the following notation. A random variable X has a LN distribution with scale parameter $\mu \in \mathbb{R}$ and shape parameter $\sigma > 0$, if its probability density function (pdf) has the form

$$g(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\}, \ x > 0. \tag{1}$$

The cumulative distribution function (cdf) of the LN distribution is easily expressed in terms of the standard normal cumulative function as

$$G(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right). \tag{2}$$

The hazard rate function corresponding to (1) is

$$h(x) = \frac{1}{x\sigma\sqrt{2\pi}\left\{1 - \Phi(\frac{\log x - \mu}{\sigma})\right\}} \exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\}.$$

In this article, we propose the *BLN* distribution which includes the log-normal distribution as special case. This generalization due to its flexibility seems be an important model that can be used in a variety of lifetime problems. The calculations involve some special functions, including the well-known error function defined by

$$\mathrm{erf(x)} = \frac{2}{\sqrt{\pi}} \int_0^{\mathrm{x}} \exp(-\mathrm{t}^2)\mathrm{dt},$$

the incomplete beta function ratio, i.e. the cdf of the beta distribution with parameters $a > 0$ and $b > 0$ given by

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1}dt,$$

the beta function defined by ($\Gamma(\cdot)$ is the gamma function)

$$B(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1}dw = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

the well-known hypergeometric function (Gradshteyn and Ryzhik, 2000) defined by (for $\alpha_k > 0$, $\beta_k > 0$, $k = 1, 2, \cdots$)

$${}_pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; x) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{x^k}{k!},$$

where $(\alpha)_i = \alpha(\alpha+1)\ldots(\alpha+i-1)$ is the ascending factorial. An important particular case corresponds to $p = 2$ and $q = 1$ and leads to ${}_2F_1(\alpha, \beta; \gamma; x)$, whereas $p = q = 1$ yields the confluent hypergeometric function ${}_1F_1(\alpha; \beta; x)$.

The rest of the paper is organized as follows. In Section 2, we define the *BLN* distribution. Probability weighted moments (PWMs) are expectations of certain functions of a random variable defined when the ordinary moments of the random variable exist. In Section 3, we derive the PWMs of the *LN* distribution. Section 4 provides a general expansion for the moments of the *BLN* distribution. Its moment generating function (mgf) is derived in Section 5. Section 6 is devoted to the characteristic function. Mean deviations are obtained in Section 7. In Sections 8 and 9, we present the Bonferroni and Lorenz curves and the entropy, respectively. Section 10 provides expansions for the *BLN* order statistics. We derive, in Section 11, expansions for their moments and for the *L*-moments. These quantities are defined by Hosking (1990) as expectations of certain linear combinations of order statistics. In Section 12, we discuss maximum likelihood estimation and calculate the elements of the observed information matrix. One application to a real data set in Section 13 illustrates the importance of the *BLN* distribution. Finally, concluding remarks are given in Section 14.

2 The New Model

The generalization of the LN distribution is motivated by the work of Eugene et al. (2002). One major benefit of the class of beta generalized distributions is its ability of fitting skewed data that can not be properly fitted by existing distributions. Consider starting from a parent cumulative function $G(x)$, they defined a class of generalized beta distributions by

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} \omega^{a-1}(1-\omega)^{b-1} d\omega = I_{G(x)}(a,b), \tag{3}$$

where $a > 0$ and $b > 0$ are two additional parameters whose role is to introduce skewness and to vary tail weight. The cdf $G(x)$ could be quite arbitrary and $F(x)$ is refereed to the beta G distribution. If V has a beta distribution with parameters a and b, application of $X = G^{-1}(V)$ yields X with cumulative distribution (3).

We can express (3) in terms of the hypergeometric function, since the properties of this function are well established in the literature. We have

$$F(x) = \frac{G(x)^a}{a\,B(a,b)}\,{}_2F_1(a, 1-b, a+1; G(x)).$$

Some generalized beta distributions were discussed in recent years. Eugene et al. (2002), Nadarajah and Kotz (2004), Nadarajah and Gupta (2004) and Nadarajah and Kotz (2005) proposed the beta normal, beta Gumbel, beta Fréchet and beta exponential distributions, respectively.

The density function corresponding to (3) can be expressed as

$$f(x) = \frac{g(x)}{B(a,b)} G(x)^{a-1}\{1 - G(x)\}^{b-1},$$

where $g(x) = dG(x)/dx$ is the density of the parent distribution. The density $f(x)$ will be most tractable when both functions $G(x)$ and $g(x)$ have simple analytic expressions. Except for some special choices of these functions, the density $f(x)$ will be difficult to deal with in generality.

The BLN density function, say $BLN(\mu, \sigma, a, b)$, with four parameters μ, σ, a and b is given by

$$f(x) = \frac{\exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\}}{x\sigma\sqrt{2\pi}B(a,b)} \Phi\left(\frac{\log x - \mu}{\sigma}\right)^{a-1} \left\{1 - \Phi\left(\frac{\log x - \mu}{\sigma}\right)\right\}^{b-1}. \tag{4}$$

Evidently, the density function (4) does not involve any complicated function but generalizes a few known distributions. The BLN distribution has a few distributions as special cases. The LN distribution arises as the particular case for $a = b = 1$. It is clear that the BLN distribution is much more flexible than the LN distribution. If $b = 1$, it leads to a new distribution, refereed to as the exponentiated log-normal (ELN) distribution. The BLN distribution is easily simulated as follows: if V has a beta distribution with parameters a and b, then $X = \exp(\sigma\Phi^{-1}(V) + \mu)$ has the $BLN(\mu, \sigma, a, b)$ distribution.

The cdf and hazard rate function corresponding to (4) are given by

$$F(x) = I_{\left[\Phi\left(\frac{\log x-\mu}{\sigma}\right)\right]}(a, b) \tag{5}$$

and

$$h(x) = \frac{\exp\left\{-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2\right\}\Phi\left(\frac{\log x-\mu}{\sigma}\right)^{a-1}\left\{1-\Phi\left(\frac{\log x-\mu}{\sigma}\right)\right\}^{b-1}}{x\sigma\sqrt{2\pi}B(a,b)\left\{1-I_{\left[\Phi\left(\frac{\log x-\mu}{\sigma}\right)\right]}(a,b)\right\}}, \tag{6}$$

respectively.

Plots of the density (4), cumulative distribution (5) and hazard rate function (6) for selected parameter values are displayed in Figures 1, 2 and 3, respectively.

3 Probability Weighted Moments

First proposed by Greenwood et al. (1979), PWMs are expectations of certain functions of a random variable whose mean exists. A general theory for PWMs covers the summarization and description of theoretical probability distributions and observed data samples, nonparametric estimation of the underlying distribution of an observed sample, estimation of parameters, quantiles of probability distributions and hypothesis tests. The PWM method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. We calculate the PWMs of the LN distribution since they are required to obtain the ordinary moments of the BLN distribution.

The PWMs of the LN distribution are formally defined by

$$\tau_{s,r} = \int_0^\infty x^s G(x)^r g(x) dx.$$

Equations (1) and (2) lead to

$$\tau_{s,r} = \int_0^\infty x^s \Phi\left(\frac{\log x - \mu}{\sigma}\right)^r \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\} dx.$$

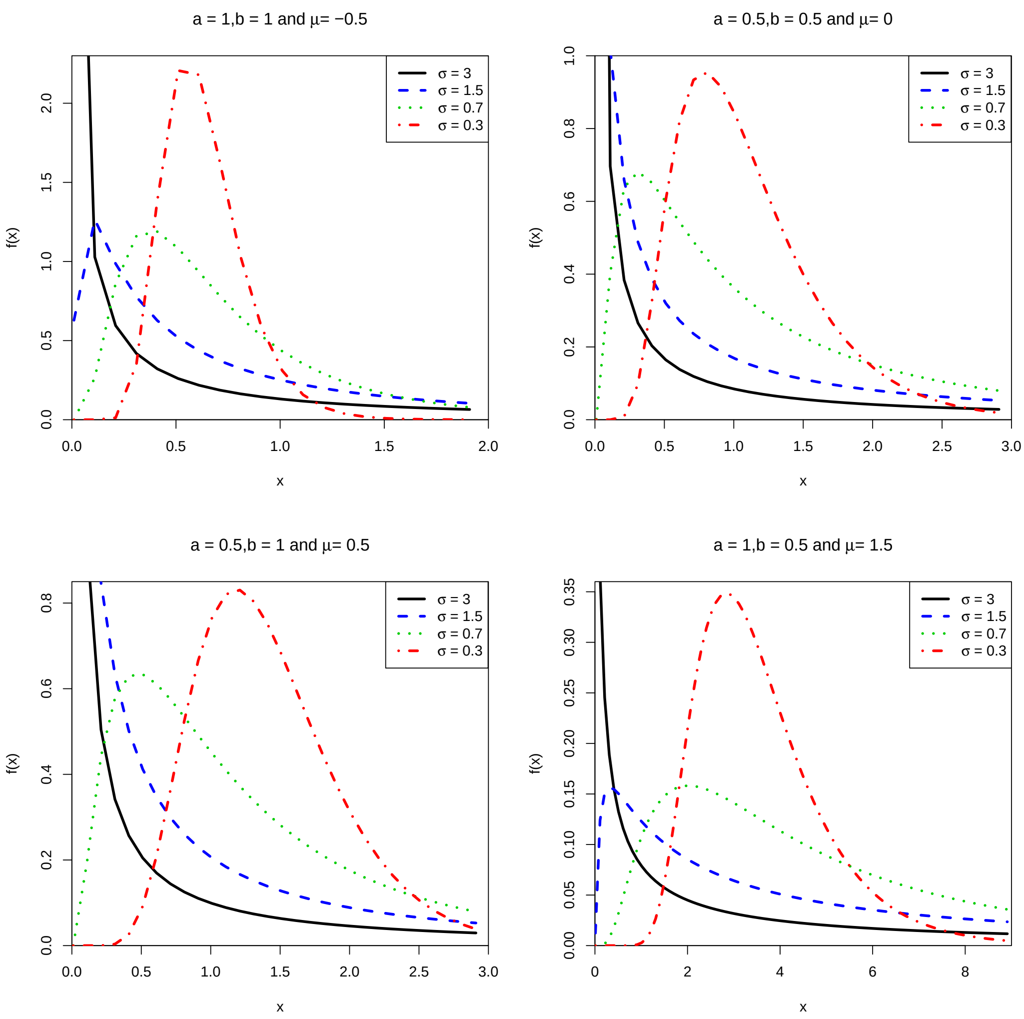

Figure 1: Plots of the BLN density (4) for selected parameter values.

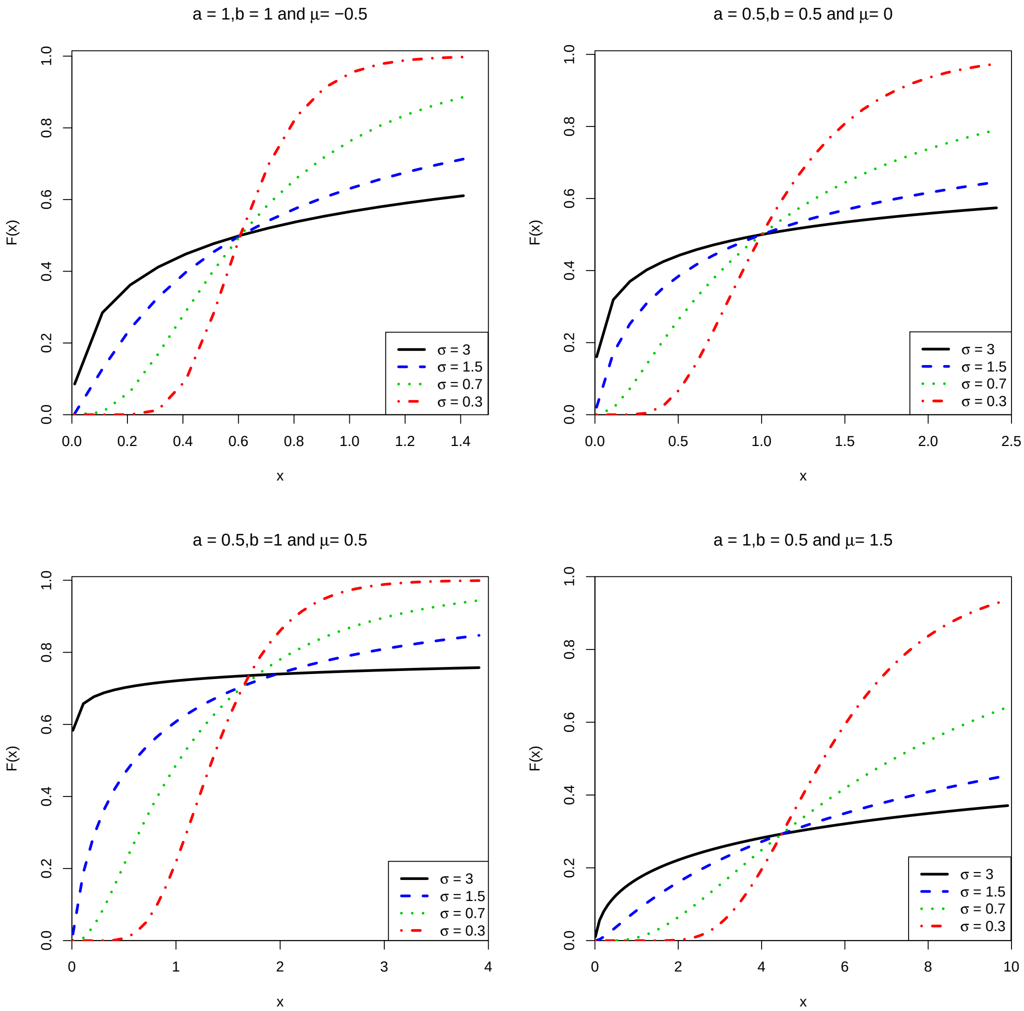

Figure 2: Plots of the BLN cumulative function (5) for selected parameter values.

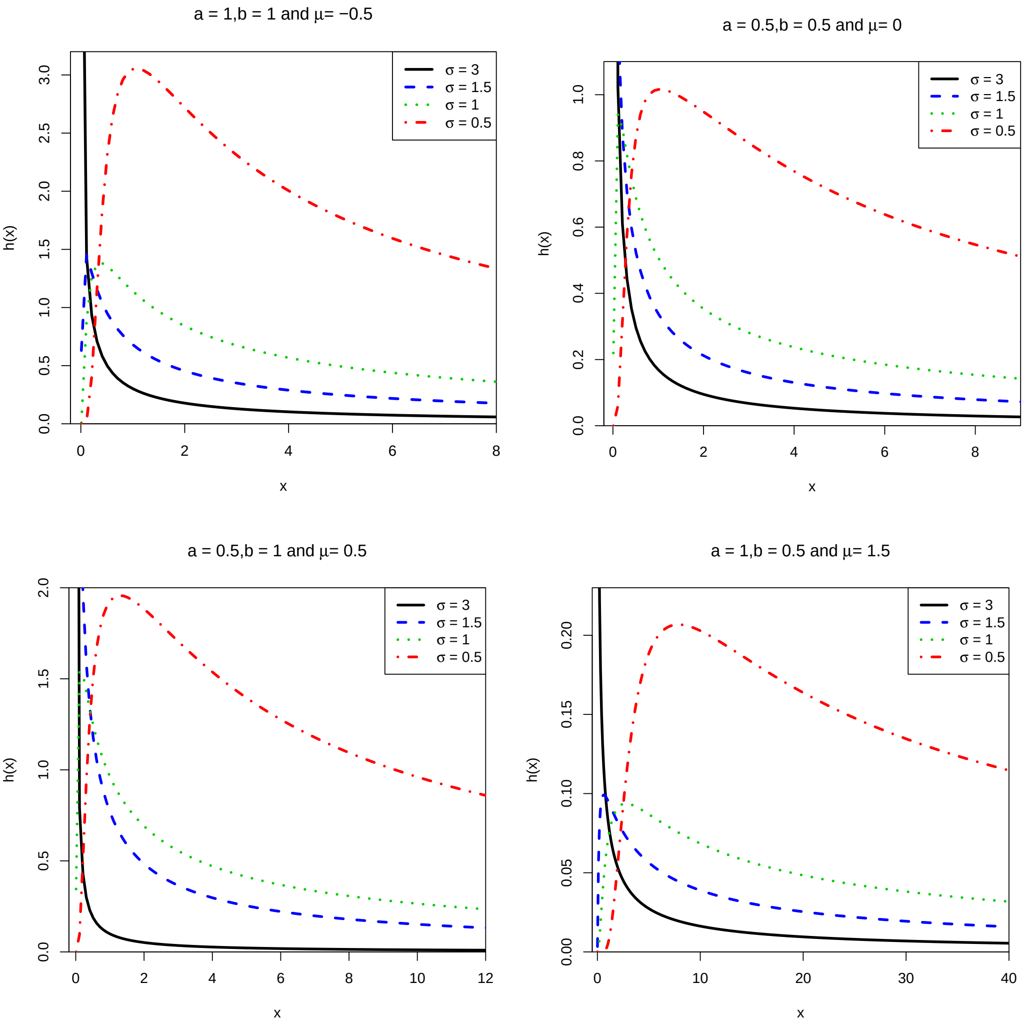

Figure 3: Plots of the BLN hazard rate function (6) for selected parameter values.

Setting $y = \dfrac{\log x - \mu}{\sigma}$, the last integral reduces to

$$\tau_{s,r} = \frac{e^{s\mu + \frac{s^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(y)^r \exp\left\{-\frac{1}{2}(y - s\sigma)^2\right\} dy. \tag{7}$$

First, we obtain

$$\Phi(y)^r = \frac{1}{2^r}\left\{1 + \mathrm{erf}\left(\frac{\mathrm{y}}{\sqrt{2}}\right)\right\}^{\mathrm{r}}.$$

Thus, the binomial expansion implies

$$\Phi(y)^r = \frac{1}{2^r}\sum_{j=0}^{r}\binom{r}{j}\mathrm{erf}\left(\frac{\mathrm{y}}{\sqrt{2}}\right)^{\mathrm{j}}.$$

From the series expansion for the error function erf(.)

$$\mathrm{erf(x)} = \frac{2}{\sqrt{\pi}}\sum_{\mathrm{k}=0}^{\infty}\frac{(-1)^{\mathrm{k}}\mathrm{x}^{2\mathrm{k}+1}}{(2\mathrm{k}+1)\mathrm{k}!}, \tag{8}$$

the last equation becomes

$$\Phi(y)^r = \frac{1}{2^r}\sum_{j=0}^{r}\binom{r}{j}\left\{\sum_{k=0}^{\infty} a_k y^{2k+1}\right\}^j,$$

where the coefficients a_k are given by $a_k = \dfrac{(-1)^k 2^{(1-2k)/2}}{\sqrt{\pi}(2k+1)k!}$. Hence,

$$\Phi(y)^r = \frac{1}{2^r}\sum_{j=0}^{r}\binom{r}{j}\sum_{k_1,\ldots,k_j=0}^{\infty} A(k_1, ..., k_j)y^{2s_j+j},$$

with $A(k_1, \ldots, k_j) = a_{k_1} \ldots a_{k_j}$ and $s_j = k_1 + \cdots + k_j$.

Inserting the preceding equation into (7) and interchanging terms, we obtain

$$\tau_{s,r} = \frac{e^{s\mu + \frac{s^2\sigma^2}{2}}}{\sqrt{2\pi}\,2^r}\sum_{j=0}^{r}\sum_{k_1,\ldots,k_j=0}^{\infty}\binom{r}{j}A(k_1, ..., k_j)K(2s_j + j), \tag{9}$$

where $K(p)$ is the integral defined by

$$K(p) = \int_{-\infty}^{\infty} y^p \exp\left\{-\frac{1}{2}(y - s\sigma)^2\right\} dy.$$

Setting $t = y - s\sigma$, $K(p)$ reduces to

$$K(p) = \int_{-\infty}^{\infty} (t + s\sigma)^p \exp\left(-\frac{t^2}{2}\right) dt.$$

Using the binomial expansion and interchanging terms, it becomes

$$K(p) = \sum_{l=0}^{p} \binom{p}{l} (s\sigma)^{p-l} (\sqrt{2\pi}) \int_{-\infty}^{\infty} t^l \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right).$$

The preceding integral is the lth moment of the standard normal random variable. Thus,

$$K(p) = \sum_{l=0}^{p} \binom{p}{l} (s\sigma)^{p-l} \frac{\sqrt{2\pi}\, l!}{2^{l/2}(l/2)!} \delta_{\{2,4,6,\cdots\}}(l),$$

where

$$\delta_A(l) = \begin{cases} 1 & \text{if } l \in A \\ 0 & \text{if } l \notin A \end{cases}$$

Inserting $K(p)$ into (9), yields

$$\begin{aligned} \tau_{s,r} &= \frac{e^{s\mu + \frac{s^2\sigma^2}{2}}}{2^r} \sum_{j=0}^{r} \sum_{k_1,\ldots,k_j=0}^{\infty} \sum_{l=0}^{2s_j+j} A(k_1, \ldots, k_j) \binom{r}{j} \binom{2s_j + j}{l} \\ &\quad \times (s\sigma)^{2s_j+j-l} \frac{l!}{2^{l/2}(l/2)!} \delta_{\{2,4,6,\cdots\}}(l). \qquad (10) \end{aligned}$$

We now give an alternative expression for the PWMs. From equation (7), we can write

$$\tau_{s,r} = e^{s\mu + s^2\sigma^2} \int_{-\infty}^{\infty} \exp(s\sigma y) \Phi(y)^r \phi(y) dy$$

and then

$$\tau_{s,r} = \frac{e^{s\mu + s^2\sigma^2} 2^{-r}}{\sqrt{2\pi}} \sum_{j=0}^{r} \binom{r}{j} \int_{-\infty}^{\infty} \exp(s\sigma y) \text{erf}\left(\frac{y}{\sqrt{2}}\right)^j \exp\left(-\frac{y^2}{2}\right) dy.$$

Using (8), we can calculate the preceding integral, say $J(s\sigma, j)$, following the same steps described by Nadarajah (2008). We have

$$
\begin{aligned}
J(s\sigma, j) &= \int_{-\infty}^{\infty} \exp(s\sigma y) \left\{ \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m+1}}{2^{m+1/2}(2m+1)m!} \right\}^j \exp\left(-\frac{y^2}{2}\right) dy \\
&= \left(\frac{2}{\sqrt{\pi}}\right)^j \sum_{m_1=0}^{\infty} \cdots \sum_{m_j=0}^{\infty} \frac{(-1)^{m.}}{2^{m.+j/2}(2m_1+1)\ldots(2m_j+1)m_1!\ldots m_j!} \\
&\times \int_{-\infty}^{\infty} y^{2m.+j} \exp\left(s\sigma y - \frac{y^2}{2}\right) dy \\
&= \left(\frac{2}{\sqrt{\pi}}\right)^j \sum_{m_1=0}^{\infty} \cdots \sum_{m_j=0}^{\infty} \frac{(-1)^{m.} M(j)}{2^{m.+j/2}(2m_1+1)\ldots(2m_j+1)m_1!\ldots m_j!},
\end{aligned}
$$

where $m. = m_1 + \ldots + m_j$. Elementary integration using equation (2.3.15.10) given by Prudnikov *et al.* (1986) yields

$$
M(j) = (-1)^{m.} i^j 2^{-(m.+j/2)} \sqrt{2\pi} \exp\left(\frac{s^2\sigma^2}{2}\right) H_{2m.+j}\left(-\frac{is\sigma}{\sqrt{2}}\right), \quad (11)
$$

where $i = \sqrt{-1}$ is the complex unit and $H_\nu(\cdot)$ denotes the Hermite polynomial of order ν. Using a result due to Withers (1999), we have

$$
H_n(x) = E\,(x + iZ)^n,
$$

where Z is a standard normal random variable. Hence, we can rewrite (11) as

$$
M(j) = (-1)^j 2^{-(m.+j/2)} \sqrt{2\pi} \exp\left(\frac{s^2\sigma^2}{2}\right) E\left(Z - \frac{s\sigma}{\sqrt{2}}\right)^{2m.+j}.
$$

Further, following the same algebraic developments by Nadarajah (2008), we obtain

$$
\begin{aligned}
J(s\sigma, j) &= 2^j \sqrt{2\pi} \left(-\frac{1}{\sqrt{\pi}}\right)^j \exp\left(\frac{s^2\sigma^2}{2}\right) \\
&\times E\Big[T^j F_A^{(j)}\Big(; \frac{1}{2}, \ldots, \frac{1}{2}; \frac{3}{2}, \ldots, \frac{3}{2}; -T^2, \ldots, -T^2\Big], \quad (12)
\end{aligned}
$$

where $T = \dfrac{Z\sqrt{2} - s\sigma}{2\sqrt{2}}$ is a linear function of the standard normal random variable. Equation (12), except for the expectation with respect to T, is a finite sum of the Lauricella function of type A (Exton, 1976). The first argument in this function is empty. The calculation of the expectation in (12) will require numerical integration and this can be performed easily because most packages have routines for the standard normal distribution. Finally, we have

$$\tau_{s,r} = \frac{e^{s\mu+s^2\sigma^2} 2^{-r}}{\sqrt{2\pi}} \sum_{j=0}^{r} \binom{r}{j} J(s\sigma, j). \qquad (13)$$

Equations (10) and (12)-(13) for the PWMs of the LN distribution are the main result of this section.

4 Moments

The cdf $F(x)$ and pdf $f(x)$ of the beta G distribution are usually straightforward to compute numerically from the baseline functions $G(x)$ and $g(x)$ and equations (3) and (2) using statistical software with numerical facilities. Here, we provide expansions for these functions in terms of infinite (or finite) weighted sums of powers of $G(x)$ which will prove useful in our case that $G(x)$ does not have a simple expression. In subsequent sections, we use these expansions to obtain formal expressions for the moments of the BLN distribution and for the density of the order statistics and their moments.

For $b > 0$ real non-integer and $a > 0$ integer, the cumulative distribution of any beta G distribution can be written as (Cordeiro and Nadarajah, 2010)

$$F(x) = \frac{1}{B(a,b)} \sum_{r=0}^{\infty} w_r G(x)^{a+r}, \qquad (14)$$

where

$$w_j = \frac{(-1)^j \binom{b-1}{j}}{(a+j)}.$$

Equation (14) gives the cdf of the beta G distribution as an infinite sum of powers of $G(x)$. Otherwise, if a is real non-integer, we can expand $G(x)^{a+j}$ from equation (42) in the Appendix, and therefore the cumulative function $F(x)$ can be expressed as a power series expansion of the baseline $G(x)$

$$F(x) = \frac{1}{B(a,b)} \sum_{r=0}^{\infty} t_r G(x)^r, \qquad (15)$$

where

$$t_r = \sum_{l=0}^{\infty} w_l s_r(a+l),$$

where the quantities $s_r(a+l)$ are easily determined from equation (43) in the Appendix. Expansions for the density function of the beta G distribution are immediately derived by simple differentiation of equations (14) and (15) for $a > 0$ integer and $a > 0$ real non-integer, respectively. We have

$$f(x) = \frac{g(x)}{B(a,b)} \sum_{r=0}^{\infty} (a+r)\, w_r G(x)^{a+r-1} \tag{16}$$

and

$$f(x) = \frac{g(x)}{B(a,b)} \sum_{r=0}^{\infty} (r+1)\, t_{r+1} G(x)^{r}. \tag{17}$$

The sth moment of the beta G distribution can then be written as an infinite sum of convenient PWMs of the baseline distribution G. These expansions are readily computed numerically using standard statistical software. They (and other expansions in the paper) can also be evaluated in symbolic computation software such as Mathematica and Maple. These symbolic software have currently the ability to deal with analytic expressions of formidable size and complexity. In numerical applications, a large natural number N can be used in the sums instead of infinity.

For a integer, equation (16) yields

$$E(X^s) = \sum_{r=0}^{\infty} \frac{(a+r)w_r}{B(a,b)} \tau_{s,a+r-1}, \tag{18}$$

whereas for a real non-integer, equation (17) implies

$$E(X^s) \;=\; \sum_{r=0}^{\infty} \frac{(r+1)t_{r+1}}{B(a,b)} \tau_{s,r}, \tag{19}$$

where $\tau_{s,r}$ can be obtained from equations (10) and (13). Expansions (18) and (19) are the main results of this section. From these expansions, the moments of the BLN distribution follow as infinite sums of certain PWMs of the LN distribution.

Let $\mu = 0$ and $\sigma = 1$. Tables 1 and 2 give some numerical values for the ordinary moments ($\mu'_r, r = 1, \ldots, 6$) and variance, skewness and kurtosis, respectively, of the BLN distribution computed using in-built functions in Mathematica. Plots of the skewness and kurtosis as functions of a and b by fixing the other parameter ($a = 2.5$ and $b = 3.5$) are given in Figure 4. These plots indicate that the skewness and kurtosis of the BLN distribution decrease with anyone shape parameter, the other parameter fixed at the above values.

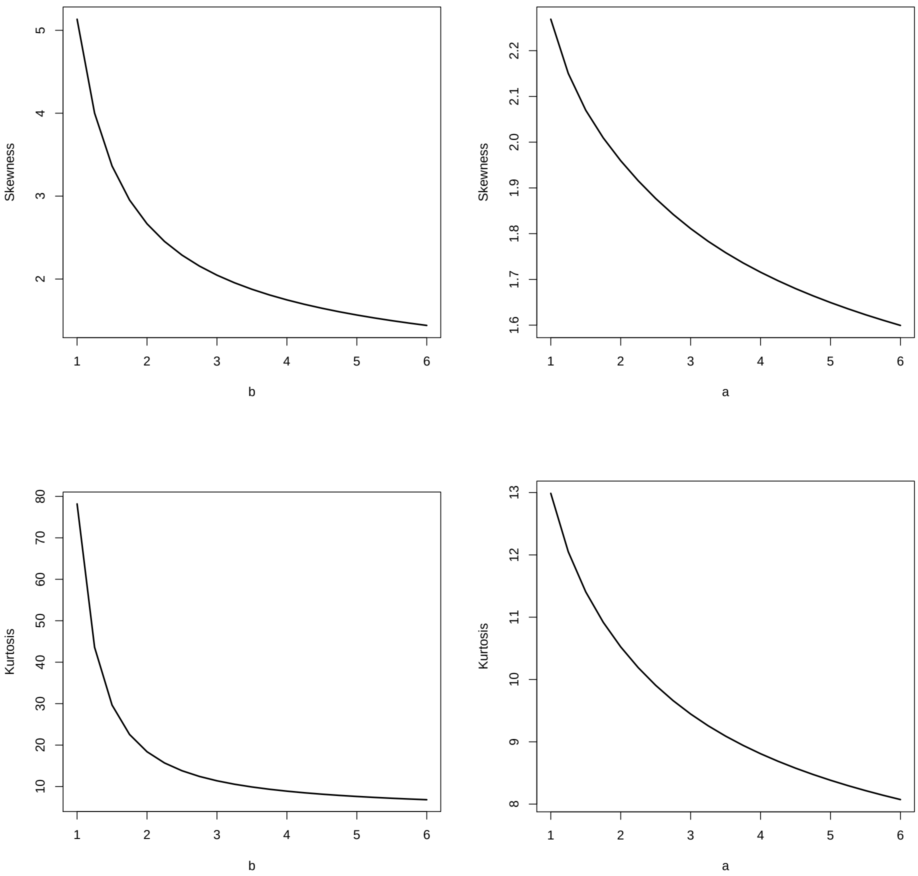

Figure 4: Skewness and kurtosis of the BLN distribution for some values of parameters a and b.

Table 1: The first six moments of $BLN(a, b, 0, 1)$ for different values a and b.

BLN	μ'_1	μ'_2	μ'_3	μ'_4	μ'_5	μ'_6
$(0.3, 0.9, 0, 1)$	0.81319	3.55409	56.24813	2.77×10^3	3.81×10^5	1.14×10^8
$(1.0, 1.0, 0, 1)$	1.64872	7.38905	90.01294	2.97×10^3	2.61×10^5	5.37×10^7
$(1.0, 1.5, 0, 1)$	1.04500	2.30679	10.43862	95.11412	1.73×10^3	6.20×10^4
$(1.0, 3.5, 0, 1)$	0.49715	0.38769	0.45208	0.76424	1.83445	6.16315
$(1.5, 1.5, 0, 1)$	1.36544	3.49437	16.98776	159.39160	2.92×10^3	1.05×10^5
$(1.5, 2.5, 0, 1)$	0.85360	1.13981	2.34578	7.38096	35.37214	257.78310
$(2.5, 3.5, 0, 1)$	0.90929	1.10097	1.76887	3.76700	10.64239	39.97635

Table 2: Variance, Skewness and kurtosis of $BLN(a, b, 0, 1)$ for different values a and b.

BLN	Variance	Skewness	Kurtosis
$(0.3, 0.9, 0, 1)$	2.89281	9.88845	311.16571
$(1.0, 1.0, 0, 1)$	4.67077	6.18443	113.68915
$(1.0, 1.5, 0, 1)$	1.21477	4.09976	42.70447
$(1.0, 3.5, 0, 1)$	0.14053	2.27054	13.00738
$(1.5, 1.5, 0, 1)$	1.62994	3.73167	35.86031
$(1.5, 2.5, 0, 1)$	0.41117	2.54457	16.33615
$(2.5, 3.5, 0, 1)$	0.27416	1.87520	9.90132

5 Moment generating function

The moment generating function (mgf) of a random variable X with density (4), say $M_X(t, \mu, \sigma) = E[\exp(tX)]$, $\forall t \in \mathbb{R}$, corresponding to $a > 0$ real non-integer, is obtained from (17) as

$$M_X(t, \mu, \sigma) = \sum_{r=0}^{\infty} \frac{(r+1)t_{r+1}}{B(a, b)} \int_0^{\infty} \exp(tx)\, G(x)^r g(x) dx.$$

We define

$$J(t, r; \mu, \sigma) = \int_0^{\infty} \exp(tx)\, G(x)^r g(x) dx. \tag{20}$$

We can verify that $J(t, r; \mu, \sigma)$ diverges for t real positive and converges for $t \in (-\infty, 0]$. From this fact, setting $t = -s$, where $s > 0$, yields

$$M_X(-s, \mu, \sigma) = \sum_{r=0}^{\infty} \frac{(r+1)t_{r+1}}{B(a, b)} J(-s, r; \mu, \sigma), \quad s > 0. \tag{21}$$

Similarly, for $a > 0$ integer, the mgf is obtained from equation (16) as

$$M_X(-s, \mu, \sigma) = \sum_{r=0}^{\infty} \frac{(a+r)w_r}{B(a,b)} J(-s, r+a-1; \mu, \sigma), \quad s > 0. \tag{22}$$

From equation (20), we have

$$\begin{aligned} J(-s, r; \mu, \sigma) &= \int_0^{\infty} \exp(-sx)\, \Phi\left(\frac{\log x - \mu}{\sigma}\right)^r \\ &\quad \times \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\} dx. \end{aligned}$$

Setting $y = \dfrac{\log x - \mu}{\sigma}$, the preceding integral can be rewritten as

$$J(-s, r; \mu, \sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-se^{\mu} \exp(\sigma y)\} \Phi(y)^r \exp\left(-\frac{y^2}{2}\right) dy.$$

By combining equations (2) and (9), we have

$$J(-s, r; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\, 2^r} \sum_{j=0}^{r} \sum_{k_1, \ldots, k_j = 0}^{\infty} \binom{r}{j} A(k_1, \ldots, k_j) I(-s, 2s_j + j; \mu, \sigma), \tag{23}$$

where s_j and $A(k_1, \ldots, k_j)$ were defined in Section 3 and $I(-s, p; \mu, \sigma)$ is

$$I(-s, p; \mu, \sigma) = \int_{-\infty}^{\infty} y^p \exp(-s \exp(\mu) \exp(\sigma y)) \exp\left(-\frac{y^2}{2}\right) dy, \quad s > 0.$$

Using

$$\exp\{-s \exp(\mu + \sigma y)\} = \exp\{-\exp(\sigma y + \mu + \log s)\}, \qquad \forall s > 0.$$

and setting $z = \sigma y + \mu + \log s$, the last integral can be expressed as

$$\begin{aligned} I(-s, p; \mu, \sigma) &= \frac{1}{\sigma^{p+1}} \int_{-\infty}^{\infty} \frac{1}{\exp(\exp(z))}\ (z - \mu - \log s)^p \\ &\quad \times \exp\left\{-\frac{1}{2}\left\{\frac{z - \mu - \log s}{\sigma}\right)^2\right\} dz. \end{aligned}$$

Using the Maclaurin series for the exponential function twice, we have

$$\exp(e^z) = e \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \tag{24}$$

where

$$B_n = e^{-1} \sum_{s=0}^{\infty} \frac{s^n}{s!}$$

are the Bell numbers. This formula is sometimes called Dobiński formula (Harris et al., 2008). The first Bell numbers are $\{1, 1, 2, 5, 15, 52, \cdots\}$ and the expansion has the form

$$\exp(e^z) = e(1 + z + z^2 + \frac{5}{6}z^3 + \frac{15}{24}z^4 + \frac{52}{120}z^5 + \cdots).$$

In order to compute the integral $I(-s, p; \mu, \sigma)$, it is sufficient to compute the inverse of the series (24). We now use equation (0.313) from Gradstheyn and Ryszhik (2000) given by

$$\frac{\sum_{k=0}^{\infty} a_k x^k}{\sum_{k=0}^{\infty} b_k x^k} = b_0^{-1} \sum_{k=0}^{\infty} c_k x^k,$$

where the coefficients c_k are easily obtained from the recurrence relation

$$c_k = a_k - b_0^{-1} \sum_{i=1}^{k} c_{k-i} b_i.$$

Here, $a_0 = 1$ and $a_k = 0, \forall k \geq 1$ and $b_n = \dfrac{B_n}{n!}, \forall n \geq 0$. Note that $b_0 = 1$. Hence,

$$\frac{1}{\exp(e^z)} = e^{-1} \sum_{n=0}^{\infty} C_n z^n, \tag{25}$$

so that C_n is defined recursively by

$$C_0 = 1 \quad \text{and} \quad C_n = -\sum_{k=1}^{n} C_{n-k} \frac{B_k}{k!}, \;\; n \geq 1.$$

The first terms of this series are given by

$$\frac{1}{\exp(e^z)} = e^{-1}(1 - z + \frac{1}{6}z^3 + \frac{1}{24}z^4 - \frac{2}{120}z^5 + \cdots).$$

Inserting (25) in the integral $I(-s, r; \mu, \sigma)$, it follows that

$$\begin{aligned} I(-s, p; \mu, \sigma) \;\; &= \;\; \frac{1}{\sigma^{p+1}} \int_{-\infty}^{\infty} e^{-1} \sum_{n=0}^{\infty} C_n z^n \;\; (z - \mu - \log s)^p \\ &\quad \times \exp\left\{ -\frac{1}{2} \left\{ \frac{z - \mu - \log s}{\sigma} \right)^2 \right\} dz. \end{aligned}$$

Using the binomial expansion and interchange terms, it becomes

$$I(-s,p;\mu,\sigma)=\frac{1}{\sigma^{p+1}}\sum_{n=0}^{\infty}\sum_{k=0}^{p}e^{-1}C_n\binom{p}{k}(-1)^{p-k}(\mu+\log s)^{p-k}\,T(-s,k+n;\mu,\sigma),$$

where $T(-s,p;\mu,\sigma)$ is defined by

$$T(-s,p;\mu,\sigma)=\int_{-\infty}^{\infty}z^p\exp\left\{-\frac{1}{2}\left(\frac{z-\mu-\log s}{\sigma}\right)^2\right\}dz.$$

Setting $x=\dfrac{z-\mu-\log s}{\sigma}$, $T(-s,p;\mu,\sigma)$ reduces to

$$T(-s,p;\mu,\sigma)=\int_{-\infty}^{\infty}(\sigma x+\mu+\log s)^p\exp\left(-\frac{x^2}{2}\right)\sigma dx,$$

and using the binomial expansion

$$T(-s,p;\mu,\sigma)=\sqrt{2\pi}\sum_{l=0}^{p}\binom{p}{l}\sigma^{l+1}(\mu+\log s)^{p-l}\int_{-\infty}^{\infty}x^l\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx.$$

The last integral is the lth moment of the normal random variable. So,

$$T(-s,p;\mu,\sigma)=\sum_{l=0}^{p}\binom{p}{l}\sigma^{l+1}(\mu+\log s)^{p-l}\frac{\sqrt{2\pi}\,l!}{2^{l/2}(l/2)!}\delta_{\{2,4,6,\cdots\}}(l).$$

Inserting the integral $I(-s,p;\mu,\sigma)$ into (23), we obtain

$$J(-s,r;\mu,\sigma)=\frac{e^{-1}}{\sqrt{2\pi}\,2^r}\sum_{j=0}^{r}\sum_{k_1,\ldots,k_j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{2s_j+j}\frac{(-1)^{2s_j+j-k}\,C_n}{\sigma^{2s_j+j+1}}\binom{r}{j}$$
$$\times\binom{2s_j+j}{k}A(k_1,...,k_j)(\mu+\log s)^{2s_j+j-k}\;T(-s,k+n;\mu,\sigma).\qquad(26)$$

Hence, the mgf follows from equations (21), (22) and (26), which are the main results of this section.

6 Characteristic Function

The characteristic function (chr) of X, say $\phi(t,\mu,\sigma)=E[\exp(itX)]$, corresponding to (21) is obtained for $t<0$ from (17) as

$$\phi_X(t,\mu,\sigma)=\sum_{r=0}^{\infty}\frac{(r+1)t_{r+1}}{B(a,b)}J(it,r;\mu,\sigma),$$

where $i = \sqrt{-1}$ and $J(t, r, ; \mu, \sigma)$ is given by (26). For $a > 0$ integer, the chr of X comes from (16) as

$$\phi_X(t, \mu, \sigma) = \sum_{r=0}^{\infty} \frac{(a+r)w_r}{B(a,b)} J(it, r+a-1; \mu, \sigma).$$

7 Mean Deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. If X has the BLN distribution with cdf $F(x)$, we can derive the mean deviations about the mean $\nu = E(X)$ and about the median m from the relations

$$\delta_1 = \int_0^{\infty} |x - \nu| \, f(x)dx \quad \text{and} \quad \delta_2 = \int_0^{\infty} |x - m| \, f(x)dx.$$

respectively. The median is the solution of the non-linear equation

$$I_{\left[\frac{\Phi\left(\alpha - \frac{\beta}{m}\right)}{\Phi(\alpha)}\right]}(a, b) = 1/2.$$

Defining the integral

$$I(s) = \int_0^{s} x f(x)dx = \int_0^{s} \frac{x g(x)}{B(a,b)} G(x)^{a-1}\{1 - G(x)\}^{b-1},$$

these measures can be calculated from

$$\delta_1 = 2\nu F(\nu) - 2I(\nu) \text{ and } \delta_2 = E(X) - 2I(m), \tag{27}$$

where $F(\nu)$ is easily obtained from equation (5). We now derive formulas to obtain the integral $I(s)$. Setting

$$\rho(s, r; \mu, \sigma) = \int_0^{s} x g(x) G(x)^r dx,$$

we can obtain from equation (16) for $a > 0$ integer

$$I(\nu) = \sum_{r=0}^{\infty} \frac{(a+r)\, w_r}{B(a,b)} \rho(\nu, a+r-1; \mu, \sigma) \tag{28}$$

and from equation (17) for $a > 0$ real non-integer

$$I(\nu) = \sum_{r=0}^{\infty} \frac{(r+1)\, t_{r+1}}{B(a,b)} \rho(\nu, r; \mu, \sigma), \tag{29}$$

where

$$\rho(s,r;\mu,\sigma) = \int_0^s \frac{x}{x\sigma\sqrt{2\pi}} \Phi\left(\frac{\log x - \mu}{\sigma}\right)^r \exp\left\{-\frac{1}{2}\left\{\frac{\log x - \mu}{\sigma}\right)^2\right\} dx.$$

Setting $y = \dfrac{\log x - \mu}{\sigma}$, $\rho(s,r;\mu,\sigma)$ can be rewritten as

$$\rho(s,r;\mu,\sigma) = \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log s - \mu}{\sigma}} \Phi(y)^r \exp\left\{-\frac{(y-\sigma)^2}{2}\right\} dy.$$

By combining equations (2) and (9), we have

$$\rho(s,r;\mu,\sigma) = \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{2\pi}\,2^r} \sum_{j=0}^{r} \sum_{k_1,\dots,k_j=0}^{\infty} \binom{r}{j} A(k_1,\dots,k_j)\ \chi\left(\frac{\log s - \mu}{\sigma}, 2s_j + j; \mu, \sigma\right),$$

where s_j and $A(k_1,\ldots,k_j)$ were defined in Section 3 and $\chi(u,p;\mu,\sigma)$ is given by

$$\chi(u,p;\mu,\sigma) = \int_{-\infty}^{u} y^p \exp\left\{-\frac{1}{2}(y-\sigma)^2\right\} dy.$$

Setting $t = y - \sigma$, the last integral reduces to

$$\chi(u,p;\mu,\sigma) = \int_{-\infty}^{u-\sigma} (t+\sigma)^p \exp\left(-\frac{t^2}{2}\right) dt.$$

Using the binomial expansion and interchanging terms, it becomes

$$\chi(u,p;\mu,\sigma) = \sum_{l=0}^{p} \binom{p}{l} \sigma^{p-l} \int_{-\infty}^{u-\sigma} t^l \exp\left(-\frac{t^2}{2}\right) dt.$$

We now define

$$G(l) = \int_0^\infty x^l e^{-x^2/2} dx = 2^{(l-1)/2}\,\Gamma((l+1)/2).$$

In order to evaluate the integral in $\chi(u,p;\mu,\sigma)$, it is necessary to consider two cases. If $u - \sigma < 0$, we have

$$\int_{-\infty}^{u-\sigma} t^l \exp\left(-\frac{t^2}{2}\right) dt = (-1)^l G(l) + (-1)^{l+1} \int_0^{\sigma-u} t^l \exp\left(-\frac{t^2}{2}\right) dt.$$

If $u-\sigma>0$, we have

$$\int_{-\infty}^{u-\sigma} t^l \exp\left(-\frac{t^2}{2}\right) dt = (-1)^l G(l) + \int_0^{u-\sigma} t^l \exp\left(-\frac{t^2}{2}\right) dt.$$

Further, the integrals of the type $H(l,q) = \int_0^q x^l e^{-x^2/2} dx$ can be determined easily as (Whittaker and Watson, 1990)

$$\begin{aligned} H(l,q) &= \frac{2^{l/4+1/4} q^{l/2+1/2} e^{-q^2/4}}{(l/2+1/2)(l+3)} M_{l/4+1/4,l/4+3/4}(q^2/2) \\ &+ \frac{2^{l/4+1/4} q^{l/2-3/2} e^{-q^2/4}}{l/2+1/2} M_{l/4+5/4,l/4+3/4}(q^2/2), \end{aligned}$$

where $M_{k,m}(x)$ is the Whittaker function. This function can be expressed in terms of the confluent hypergeometric function ${}_1F_1$ (see Section 1) as $M_{k,m}(x) = e^{-x/2} x^{m+1/2} {}_1F_1(\frac{1}{2}+m-k; 1+2m; x)$. Hence, $\chi(u,p;\mu,\sigma)$ can be determined as

$$\begin{aligned} \chi(u,p;\mu,\sigma) &= \sum_{l=0}^{p} \binom{p}{l} \sigma^{p-l} \left[(-1)^l G(l) + H(l,u-\sigma)\delta_A(s)\right] \\ &+ \sum_{l=0}^{p} \binom{p}{l} \sigma^{p-l} \left[(-1)^{l+1} H(l,\sigma-u)(1-\delta_A(s))\right], \end{aligned}$$

where

$$\delta_A(s) = \delta_{\{u-\sigma>0\}}(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A. \end{cases}$$

Hence, we have all quantities to calculate $\rho(s,r;\mu,\sigma)$, $I(\nu)$ and then the mean deviations (27).

8 Bonferroni and Lorenz Measures

Bonferroni and Lorenz curves have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. For a random variable X with quantile function $F^{-1}(\cdot)$, the Bonferroni and Lorenz curves are defined (for $0 \le p \le 1$) by

$$B(p) = \frac{1}{p\nu} \int_0^{F^{-1}(p)} t f(t) dt \quad \text{and} \quad L(p) = \frac{1}{\nu} \int_0^{F^{-1}(p)} t f(t) dt,$$

respectively, where $\nu = E(X)$. Using the same integral $I(s) = \int_0^s x f(x) dx$ defined in Section 6, these measures can be calculated from

$$B(p) = \frac{1}{p\nu} I(q) \text{ and } L(p) = \frac{1}{\nu} I(q),$$

where $q = F^{-1}(p)$ and $I(q)$ can be obtained from equations (28) and (29) depending if a is integer and real non-integer, respectively.

We now give an expansion for $q = F^{-1}(p)$. First, from

$$G(q) = \Phi\left(\frac{\log q - \mu}{\sigma}\right)$$

we have

$$q = \exp\left(\sigma\Phi^{-1}(G(q)) + \mu\right). \tag{30}$$

Further,

$$p = F(q) = I_{G(q)}(a, b).$$

The following expansion for the inverse of the beta incomplete function $I_p^{-1}(a, b)$ can be found in wolfram website[1]

$$\begin{aligned} G(q) = I_p^{-1}(a,b) \quad = \quad & w + \frac{b-1}{a+1}w^2 + \frac{(b-1)(a^2+3ba-a+5b-4)}{2(a+1)^2(a+2)}w^3 \\ & + \frac{(b-1)[a^4+(6b-1)a^3+(b+2)(8b-5)a^2]}{3(a+1)^3(a+2)(a+3)}w^4 \\ & + \frac{(b-1)[(33b^2-30b+4)a+b(31b-47)+18]}{3(a+1)^3(a+2)(a+3)}w^4 \\ & + O(p^{5/a}), \end{aligned}$$

where $w = [apB(a, b)]^{1/a}$ for $a > 0$. Inserting the last expansion in equation (30) we can obtain q in terms of p.

9 Entropy

An entropy of a random variable X is a measure of variation of the uncertainty. One of the popular entropy measure is the Rényi entropy defined by

$$\Im(\gamma) = \frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)dx\right\},$$

where $\gamma > 0$ and $\gamma \neq 1$ (Rényi, 1961). For the density (4), we have

$$f^{\gamma}(x) = \left\{\frac{\exp\left\{-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2\right\}}{x\sigma\sqrt{2\pi}B(a,b)}\Phi\left(\frac{\log x-\mu}{\sigma}\right)^{a-1}\left\{1-\Phi\left(\frac{\log x-\mu}{\sigma}\right)\right\}^{b-1}\right\}^{\gamma},$$

[1]http://functions.wolfram.com/06.23.06.0004.01

and then

$$f^{\gamma}(x) = \frac{\exp\left\{-\frac{\gamma}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2\right\}}{(x\sigma\sqrt{2\pi})^{\gamma}B(a,b)^{\gamma}}\Phi\left(\frac{\log x-\mu}{\sigma}\right)^{(a-1)\gamma}\left\{1-\Phi\left(\frac{\log x-\mu}{\sigma}\right)\right\}^{(b-1)\gamma}.$$

If $(b-1)\gamma > 0$, an expansion for $f^{\gamma}(x)$ is immediately obtained as (Cordeiro and Nadarajah, 2010)

$$f^{\gamma}(x) = C(x)\sum_{r=0}^{\infty}(-1)^r\binom{(b-1)\gamma}{r}\Phi\left(\frac{\log x-\mu}{\sigma}\right)^{(a-1)\gamma+r},$$

where

$$C(x) = \frac{1}{(x\sigma\sqrt{2\pi})^{\gamma}B(a,b)^{\gamma}}\exp\left\{-\frac{\gamma}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2\right\}.$$

If $(a-1)\gamma > 0$, we can expand $\Phi(\cdot)^{(a-1)\gamma+r}$ using equation (42) given in the Appendix. Then, the function $f^{\gamma}(x)$ can be expressed as a power series of the normal cdf $\Phi(.)$

$$f^{\gamma}(x) = C(x)\sum_{r=0}^{\infty}t_r(\gamma)\Phi\left(\frac{\log x-\mu}{\sigma}\right)^{r},$$

where $t_r(\gamma)$ from now on is given by

$$t_r(\gamma) = \sum_{k=0}^{\infty}(-1)^k\binom{(b-1)\gamma}{k}s_r((a-1)\gamma+k),$$

and the quantities $s_r((a-1)\gamma+k)$ are easily determined from equation (43) (see Appendix).

Defining the integral

$$I_R(\gamma) = \int_0^{\infty}\frac{1}{(x\sigma\sqrt{2\pi})^{\gamma}}\exp\left\{-\frac{\gamma}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2\right\}\Phi\left(\frac{\log x-\mu}{\sigma}\right)^{r}dx,$$

the entropy measure can be calculated from

$$\Im(\gamma) = \frac{1}{1-\gamma}\log\left\{\sum_{r=0}^{\infty}\frac{t_r(\gamma)}{B(a,b)^{\gamma}}\,I_R(\gamma)\right\}. \quad (31)$$

Setting $y = \dfrac{\log x-\mu}{\sigma}$, $I_R(\gamma)$ reduces to

$$I_R(\gamma) = \frac{e^{\mu(1-\gamma)}}{\sigma^{(\gamma-1)}(\sqrt{2\pi})^{\gamma}}\int_{-\infty}^{\infty}\Phi(y)^r\exp\left\{-\frac{\gamma}{2}y^2+\sigma(1-\gamma)y\right\}dy.$$

By combining equations (2) and (9), $I_R(\gamma)$ can be written as

$$I_R(\gamma) = \frac{e^{\mu(1-\gamma)}}{\sigma^{\gamma-1}(\sqrt{2\pi})^{\gamma}\, 2^r} \sum_{j=0}^{r} \sum_{k_1,\ldots,k_j=0}^{\infty} \binom{r}{j} A(k_1,...,k_j) J_{\gamma}(2s_j + j, \sigma), \tag{32}$$

where s_j and $A(k_1, \ldots, k_j)$ were defined in Section 3 and $J_{\gamma}(p, \sigma)$ is given by

$$J_{\gamma}(p,\sigma) = \int_{-\infty}^{\infty} y^p \exp\left\{-\frac{\gamma}{2}y^2 + \sigma(1-\gamma)y\right\} dx.$$

We can transform this integral to

$$J_{\gamma}(p,\sigma) = e^{\frac{(\sigma(1-\gamma))^2}{2\gamma}} \int_{-\infty}^{\infty} y^p \exp\left\{-\frac{1}{2}\left(\sqrt{\gamma}y - \frac{\sigma(1-\gamma)}{\sqrt{\gamma}}\right)^2\right\} dx.$$

Setting $t = \sqrt{\gamma}y - \dfrac{\sigma(1-\gamma)}{\sqrt{\gamma}}$, $J_{\gamma}(p,\sigma)$ reduces to

$$J_{\gamma}(p,\sigma) = \frac{\sqrt{2\pi}\, e^{\frac{(\sigma(1-\gamma))^2}{2\gamma}}}{(\sqrt{\gamma})^p} \int_{-\infty}^{\infty} \left(t + \frac{\sigma(1-\gamma)}{\sqrt{\gamma}}\right)^p \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt.$$

From the binomial expansion, interchanging terms and using the moments of the normal random variable, it becomes

$$J_{\gamma}(p,\sigma) = \frac{\sqrt{2\pi}\, e^{\frac{(\sigma(1-\gamma))^2}{2\gamma}}}{(\sqrt{\gamma})^p} \sum_{l=0}^{p} \binom{p}{l} \left(\frac{\sigma(1-\gamma)}{\sqrt{\gamma}}\right)^{p-l} \frac{l!}{2^{l/2}(l/2)!} \delta_{\{2,4,6,\cdots\}}(l).$$

Inserting (9) and (32) into (31), we obtain

$$\begin{aligned}\Im(\gamma) =& \frac{1}{1-\gamma} \log\left\{\frac{e^{\mu(1-\gamma)+\frac{(\sigma(1-\gamma))^2}{2\gamma}}}{\sigma^{\gamma-1}(\sqrt{2\pi})^{\gamma-1}(\sqrt{\gamma})^p}\right\} \\ &+ \frac{1}{1-\gamma} \log\left\{\sum_{r=0}^{\infty}\sum_{j=0}^{r}\sum_{k_1,\ldots,k_j=0}^{\infty}\sum_{l=0}^{2s_j+j} \frac{t_r(\gamma)}{2^r B(a,b)^{\gamma}} A(k_1,...,k_j)\right. \\ &\left. \times \binom{r}{j}\binom{2s_j+j}{l}\left(\frac{\sigma(1-\gamma)}{\sqrt{\gamma}}\right)^{2s_j+j-l} \frac{l!}{2^{l/2}(l/2)!}\delta_{\{2,4,6,\cdots\}}\right\}.\end{aligned}$$

10 Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. The density $f_{i:n}(x)$ of the ith order statistic for $i = 1, \ldots, n$ from data values $X_1, \ldots, X_n$ having the beta G distribution is

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x) F(x)^{i-1} \{1 - F(x)\}^{n-i}$$

and then

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{i+j-1}. \qquad (33)$$

By combining (2) and (33), $f_{i:n}(x)$ becomes

$$f_{i:n}(x) = \frac{g(x)G(x)^{a-1}\{1 - G(x)\}^{b-1}}{B(a,b)B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{i+j-1}. \qquad (34)$$

We now use an equation of Gradshteyn and Ryzhik (2000, Section 0.314) for power series raised to powers. For any j positive integer, we have

$$\left(\sum_{i=0}^{\infty} w_i u^i \right)^j = \sum_{i=0}^{\infty} c_{i,j} u^i,$$

where the coefficients $c_{i,j}$ for $i = 1, 2, \ldots$ can be easily obtained from the recurrence relation

$$c_{i,j} = (i w_0)^{-1} \sum_{m=1}^{i} (jm - i + m) w_m c_{i-m,j}, \qquad (35)$$

with $c_{0,j} = w_0^j$. The coefficient $c_{i,j}$ comes from $c_{0,j}, \ldots, c_{i-1,j}$ and therefore are obtained from $w_0, \ldots, w_i$. Clearly, $c_{i,j}$ can be given explicitly in terms of the quantities w_i, although it is not necessary for programming numerically our expansions in any algebraic or numerical software.

For $a > 0$ integer, equation (14) implies that

$$F(x)^{i+j-1} = \left(\frac{G(x)^a}{B(a,b)} \right)^{i+j-1} \left(\sum_{r=0}^{\infty} w_r G(x)^r \right)^{i+j-1}.$$

For $b > 0$ real non-integer, setting $u = G(x)$, using equation (34) and then the power series for $(1-u)^{b-1}$, yields

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{r,l=0}^{\infty} (-1)^{j+l} c_{r,i+j-1} \binom{n-i}{j} \frac{g(x) G(x)^{r+l+a(i+j)-1}}{B(a,b)^{i+j} B(i, n-i+1)}, \qquad (36)$$

where the coefficients are obtained from equation (35). If b is a integer, the index l in the sum (36) stops at $b-1$.

For $a > 0$ real non-integer, equation (15) leads to

$$F(x)^{i+j-1} = \frac{1}{B(a,b)^{i+j-1}} \left(\sum_{r=0}^{\infty} t_r G(x)^r \right)^{i+j-1}.$$

In the same way, using equation (35), it follows for $b > 0$ real non-integer

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{r,l=0}^{\infty} (-1)^{j+l}\, d_{r,i+j-1} \binom{n-i}{j} \frac{g(x) G(x)^{r+l+a-1}}{B(a,b)^{i+j} B(i, n-i+1)}, \qquad (37)$$

where the coefficients are given by

$$d_{r,i+j-1} = (r t_0)^{-1} \sum_{m=1}^{r} \{(i+j)m - r\} t_m d_{r-m,i+j-1}, \qquad (38)$$

with $d_{0,i+j-1} = t_0^{i+j-1}$. If b is a integer, the index l in the sum (37) stops at $b-1$. Equations (36) and (37) are the main results of this section for $a > 0$ integer and $a > 0$ real non-integer, respectively.

11 Moments of order statistics

The sth ordinary moment of the ith order statistic, say $X_{i:n}$, for $a > 0$ integer, follows from equation (36)

$$E(X_{i:n}^s) = \sum_{j=0}^{n-i} \sum_{r,l=0}^{\infty} \frac{(-1)^{j+l} c_{r,i+j-1} \binom{n-i}{j}}{B(a,b)^{i+j} B(i, n-i+1)} \tau_{s,r+a(i+j)+l-1}, \qquad (39)$$

where the coefficient $c_{r,i+j-1}$ is defined by (35). If b is an integer, the index l in the above sum stops at $b-1$. For $a > 0$ real non-integer, equation (37) gives

$$E(X_{i:n}^s) = \sum_{j=0}^{n-i} \sum_{r,l=0}^{\infty} \frac{(-1)^{j+l} d_{r,i+j-1} \binom{n-i}{j}}{B(a,b)^{i+j} B(i, n-i+1)} \tau_{s,r+a+l-1}, \qquad (40)$$

where $d_{r,i+j-1}$ is defined by (38). If b is an integer, the index l in the above sum stops at $b-1$.

Expansions (39) and (40) are the main results of this section. The L-moments (Hosking, 1990) are analogous to the ordinary moments but can be estimated by linear functions of order statistics defined by

$$\lambda_{r+1} = r(r+1)^{-1}\sum_{k=0}^{r}\frac{(-1)^k}{k}E(X_{r+1-k:r+1}),\ r=0,1,\ldots$$

The first four L-moments are $\lambda_1 = E(X_{1:1})$, $\lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$. The L-moments have the advantage that they exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. The L-moments can be calculated from the means of the order statistics for $a > 0$ integer and $a > 0$ real non-integer by setting $s = 1$ in equations (39) and (40), respectively,

12 Estimation and Inference

Consider that X follows the BLN distribution and let $\theta = (\mu,\sigma,a,b)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\mu,\sigma,a,b)$ for a single observation x of X is

$$\begin{aligned}\ell &= \log\{\phi(t)\} - \log(x) - \log(\sigma) - \log\{B(a,b)\}\\ &+ (a-1)\log\{\Phi(t)\} + (b-1)\log\{1-\Phi(t)\},\quad x>0,\end{aligned}$$

where $t = \dfrac{\log x - \mu}{\sigma}$. The unit score vector $U = (\frac{\partial\ell}{\partial\mu}, \frac{\partial\ell}{\partial\sigma}, \frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b})^T$ has components

$$\begin{aligned}\frac{\partial\ell}{\partial\mu} &= \frac{1}{\sigma}\left[t - (a-1)\left\{\frac{\phi(t)}{\Phi(t)}\right\} + (b-1)\left\{\frac{\phi(t)}{1-\Phi(t)}\right\}\right],\\ \frac{\partial\ell}{\partial\sigma} &= \frac{1}{\sigma}\left[t^2 - 1 - (a-1)\left\{\frac{t\phi(t)}{\Phi(t)}\right\} + (b-1)\left\{\frac{t\phi(t)}{1-\Phi(t)}\right\}\right],\\ \frac{\partial\ell}{\partial a} &= \log(\Phi(t)) + \psi(a+b) - \psi(a),\\ \frac{\partial\ell}{\partial b} &= \log(1-\Phi(t)) + \psi(a+b) - \psi(b).\end{aligned}$$

The expected value of the score vector vanishes. Defining $T = \dfrac{\log X - \mu}{\sigma}$, we obtain

$$E\left\{\frac{\phi(T)}{\Phi(T)}\right\} = E\left\{\frac{\phi(T)}{1-\Phi(T)}\right\} = E\left\{\frac{T\phi(T)}{1-\Phi(T)}\right\} = E\left\{\frac{T\phi(T)}{\Phi(T)}\right\} = 0$$
$$E\left(\frac{\log X - \mu}{\sigma}\right) = (a-1)E\left\{\frac{\phi(T)}{\Phi(T)}\right\} - (b-1)E\left\{\frac{\phi(T)}{1-\Phi(T)}\right\} = 0.$$

For a random sample $x = (x_1, \ldots, x_n)$ of size n from X, the total log-likelihood is $\ell_n = \ell_n(\mu, \sigma, a, b) = \sum_{i=1}^n \ell^{(i)}$, where $\ell^{(i)}$ is the log-likelihood for the ith observation ($i = 1, \ldots, n$). The total score function is $U_n = \sum_{i=1}^n U^{(i)}$, where $U^{(i)}$ has the form given before for $i = 1, \ldots, n$. The maximum likelihood estimate (MLE) $\hat{\theta}$ of θ is obtained numerically from the nonlinear equations $U_n = 0$. For interval estimation and hypothesis tests on the parameters in θ we require the 4×4 unit information matrix

$$K = K(\theta) = \begin{bmatrix} \kappa_{\mu,\mu} & \kappa_{\mu,\sigma} & \kappa_{\mu,a} & \kappa_{\mu,b} \\ \kappa_{\sigma,\mu} & \kappa_{\sigma,\sigma} & \kappa_{\sigma,a} & \kappa_{\sigma,b} \\ \kappa_{a,\mu} & \kappa_{a,\sigma} & \kappa_{a,a} & \kappa_{a,b} \\ \kappa_{b,\mu} & \kappa_{b,\sigma} & \kappa_{b,a} & \kappa_{b,b} \end{bmatrix}.$$

The total information matrix is $K_n(\theta) = nK(\theta)$. We define the following expectations for $r = 0, 1, 2, 3$ and $s = 1, 2$

$$m_{r,s} = E\left[T^r \left\{\frac{\phi(T)}{\Phi(T)}\right\}^s\right] \quad \text{and} \quad n_{r,s} = E\left[T^r \left\{\frac{\phi(T)}{1-\Phi(T)}\right\}^s\right],$$

which can be obtained by numerical integration. The elements of the information matrix K are given by

$$\begin{aligned}
\kappa_{\mu,\mu} &= -\frac{1}{\sigma^2}\left[1 + (a-1)m_{0,2} + (b-1)n_{0,2}\right] \\
\kappa_{\mu,\sigma} &= -\frac{1}{\sigma^2}\left[(a-1)(m_{2,1} + m_{1,2}) - (b-1)(n_{2,1} - n_{1,2})\right], \\
\kappa_{\mu,a} &= 0, \quad \kappa_{\mu,b} = 0, \\
\kappa_{\sigma,\sigma} &= \frac{1}{\sigma^2}\left[2 - (a-1)(m_{3,1} + m_{2,2}) + (b-1)(n_{3,1} - n_{2,2})\right], \\
\kappa_{\sigma,a} &= 0, \quad \kappa_{\sigma,b} = 0, \quad \kappa_{a,a} = \psi'(a) - \psi'(a+b), \quad \kappa_{b,b} = \psi'(b) - \psi'(a+b), \\
\kappa_{a,b} &= -\psi'(a+b).
\end{aligned}$$

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of

$$\sqrt{n}(\hat{\theta} - \theta) \text{ is } N_4(0, K(\theta)^{-1}).$$

The asymptotic multivariate normal $N_4(0, K_n(\hat{\theta})^{-1})$ distribution of $\hat{\theta}$ can be used to construct approximate confidence intervals and confidence regions for the parameters and for the hazard and survival functions. An asymptotic confidence interval with significance level γ for each parameter θ_r is

$$ACI(\theta_r, 100(1-\gamma)\%) = (\hat{\theta_r} - z_{\gamma/2}\sqrt{\kappa^{\theta_r,\theta_r}}, \hat{\theta_r} + z_{\gamma/2}\sqrt{\kappa^{\theta_r,\theta_r}}),$$

where $\kappa^{\theta_r,\theta_r}$ is the rth diagonal element of $K_n(\theta)^{-1}$ for $r = 1, \ldots, 4$ and $z_{\gamma/2}$ is the quantile $1 - \gamma/2$ of the standard normal distribution.

The likelihood ratio (LR) statistic is useful for testing goodness of fit of the BLN distribution and for comparing this distribution with some of its special sub-models. If we consider the partition $\theta = (\theta_1^T, \theta_2^T)^T$, tests of hypotheses of the type $H_0 : \theta_1 = \theta_1^{(0)}$ versus $H_A : \theta_1 \neq \theta_1^{(0)}$ can be performed via LR tests. The LR statistic for testing the null hypothesis H_0 is $w = 2\{\ell(\hat{\theta}) - \ell(\tilde{\theta})\}$, where $\hat{\theta}$ and $\tilde{\theta}$ are the MLEs of θ under H_A and H_0, respectively. Under the null hypothesis, $w \xrightarrow{d} \chi^2_q$, where q is the dimension of the vector θ_1 of interest. The LR test rejects H_0 if $w > \xi_\gamma$, where ξ_γ denotes the upper $100\gamma\%$ point of the χ^2_q distribution. For example, we can check if the fit using the BLN distribution is statistically "superior" to a fit using the LN distribution for a given data set by testing $H_0 : a = b = 1$ versus $H_A : H_0$ is not true.

13 Application

In this section, we consider a real data set for illustrative purpose to verify how our methods work in practice. The data is obtained from McCool (1974). It represent the fatigue life in hours of 10 bearings of a certain type. These data were used as an illustrative example for the three-parameter Weibull distribution by Cohen et al. (1984) and for the two-parameter Birnbaum-Saunders (BS) distribution by Ng et al. (2003).

We computed the maximum values of the log-likelihoods to obtain LR statistics for testing nested models and generalized LR statistics for comparing non-nested models. The LR statistic for comparing the nested models H_0 : LN $\times$ H_A : BLN is $w = 2\{-53.44073 - (-56.5434)\} = 6.20534$ (p-value= 0.044). So, it follows that the BLN model fits the data significantly better that the LN distribution.

A generalized LR statistic can be used for discriminating among non-nested models as discussed in Cameron and Trived (1998, p.184). We now compare the non-nested models: Birnbaum-Saunders ($BS(\beta,\eta)$) and Weibull ($W(\beta,\eta)$) models. Then, the chi-square distribution is no longer appropriate. Consider choosing between two non-nested models F_θ and G_γ with corresponding densities functions $f(y_i|x_i,\theta)$ and $g(y_i|x_i,\gamma)$, respectively. Vuong (1989) proposed the statistic

$$\begin{aligned} T_{LR,NN} &= \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \log \frac{f(y_i|x_i,\hat{\theta})}{g(y_i|x_i,\hat{\gamma})} \right\} \\ &\times \left\{ \frac{1}{n} \sum_{i=1}^{n} \left(\log \frac{f(y_i|x_i,\hat{\theta})}{g(y_i|x_i,\hat{\gamma})} \right)^2 - \left(\frac{1}{n} \sum_{i=1}^{n} \log \frac{f(y_i|x_i,\hat{\theta})}{g(y_i|x_i,\hat{\gamma})} \right)^2 \right\}^{-1}. \end{aligned} \tag{41}$$

For strictly non-nested models, the statistic (41) converges, under the null hypothesis of equivalence of the models, to the standard normal distribution. Thus, the null hypothesis is not rejected (at significance level α) if $T_{LR,NN} \leq z_{\alpha/2}$. On the other hand, we reject the null hypothesis or the equivalence of the models in favor of the model F_θ being better (or worse) than model G_γ if $T_{LR,NN} > z_\alpha$ (or $T_{LR,NN} < -z_\alpha$).

We now compare via (41) the non-nested models fitted to these data: Weibull model versus BLN model and BS model versus BLN model. For example, let $f(y_i|x_i,\theta)$ and $g(y_i|x_i,\gamma)$ be the density functions of the BLN and Weibull distributions, respectively. Thus, the corresponding generalized LR statistics ($T_{LR,NN}$) for testing the null Weibull and BS models are equal to 1.9800 ($p-value < 0.0238$) and 2.8578 ($p-value < 0.0021$). Thus, the BLN model is significantly better than the Weibull and BS models according to the generalized LR statistics.

The MLEs of the model parameters, the maximized log-likelihoods and the values of the Akaike information criterion (AIC) and Bayesian information criterion (BIC) for the fitted models are given in Table 3. These results indicate that the BLN model has the lowest values for the AIC and BIC statistics among the fitted models, and therefore it could be chosen as the best model.

The conclusion of the LR test can be supported by means of probability plots and density plots. The probability plot consists of plots of the observed probabilities against the probabilities predict by the fitted model (see Chambers et al., 1983). For example, for each model, $F(x_{(j)})$ was plotted versus $(j-0.375)/(n+0.25)$, $j=1,\ldots,n$, where $x_{(j)}$ are the sorted observations. The probability plots for the fitted LN and BLN models are shown in Figure 5. It is clear that the BLN model yields points closer to the diagonal line.

A density plot compares the fitted densities of the models with the histogram of the observed data. This plot (given in Figure 6) shows again that the BLN model fits better to the data than the other three models.

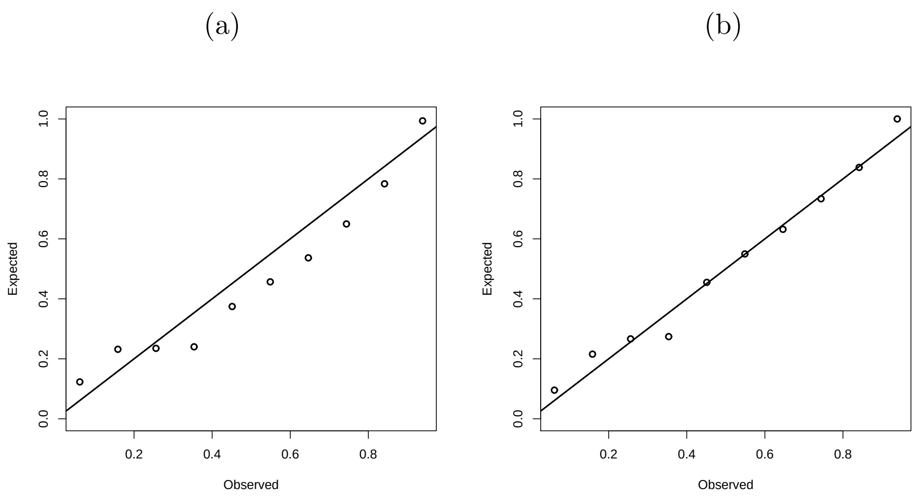

Figure 5: Probability plots of the fits of the LN distribution (a) and the BLN distribution (b) for the fatigue lifetime data.

Table 3: MLEs of the parameters for some fitted models to the fatigue lifetime data, their maximized log-likelihoods and the values of the AIC and BIC statistics.

	Estimated Parameters			
Parameter	BLN	LN	W	BS
a	8.6426	-	-	-
b	0.2847	-	-	-
μ	4.7295	5.3519	-	-
σ	0.2315	0.2788	-	-
β	-	-	2.9359	0.2825
η	-	-	246.4086	212.0501
log-likelihood	-53.4407	-56.5434	-57.3013	-54.9718
AIC	110.8815	117.0868	118.6026	113.9435
BIC	111.4866	117.6920	119.2078	114.5487

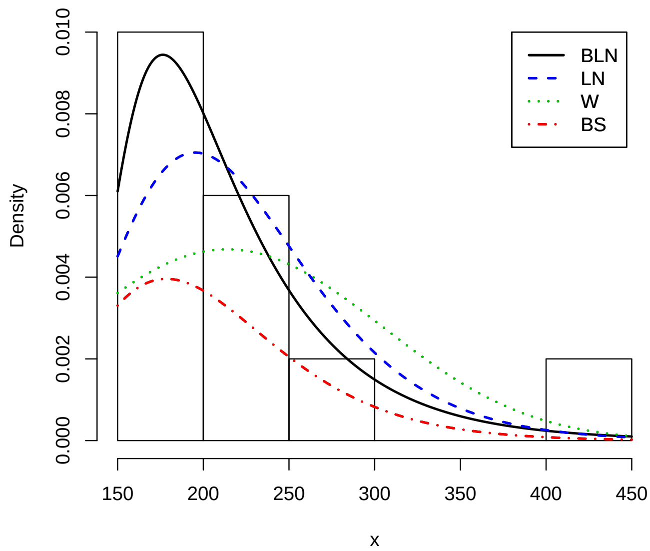

Figure 6: Histogram and estimated densities of some fitted models to the fatigue lifetime data.

14 Conclusions

In this article, we introduce the four-parameter beta log-normal (BLN) distribution that extends the log-normal distribution. This is achieved by (the well known technique) following the idea of the cumulative distribution function of the class of beta generalized distributions proposed by Eugene et al. (2002). The BLN distribution is quite flexible in analyzing positive data in place of the gamma, Weibull, Birnbaum-Saunders, generalized exponential and beta exponential distributions. It is useful to model asymmetric data to the right and unimodal distributions. We provide a mathematical treatment of the new distribution including expansions for the density and cumulative functions, moment generating function, ordinary moments, mean deviations, Bonferroni and Lorenz curves, order statistics and their moments and L-moments. The estimation of parameters is approached by the method of maximum likelihood and the expected information matrix is derived. One application of the BLN distribution shows that the new distribution could provide a better fit than other statistical models widely used in lifetime data analysis.

Appendix

We derive an expansion for $G(x)^\rho$ which holds for $\rho > 0$ real non-integer. We can write

$$G(x)^\rho = [1 - \{1 - G(x)\}]^\rho = \sum_{j=0}^{\infty} \binom{\rho}{j} (-1)^j \{1 - G(x)\}^j$$

and then

$$G(x)^\rho = \sum_{j=0}^{\infty} \sum_{r=0}^{j} (-1)^{j+r} \binom{\rho}{j} \binom{j}{r} G(x)^r.$$

We can substitute $\sum_{j=0}^{\infty} \sum_{r=0}^{j}$ for $\sum_{r=0}^{\infty} \sum_{j=r}^{\infty}$ to obtain

$$G(x)^\rho = \sum_{r=0}^{\infty} \sum_{j=r}^{\infty} (-1)^{j+r} \binom{\rho}{j} \binom{j}{r} G(x)^r.$$

and then

$$G(x)^\rho = \sum_{r=0}^{\infty} s_r(\rho) G(x)^r, \tag{42}$$

where

$$s_r(\rho) = \sum_{j=r}^{\infty} (-1)^{r+j} \binom{\rho}{j} \binom{j}{r}. \tag{43}$$

Equations (42) and (43) are used in Section 4.

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