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Exit time for a reaction diffusion model (Case of one well potential)

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We consider a interacting particle system, the Glauber +Kawasaki model. This model is the result of the combination of a fast stirring, the Kawasaki part, and a spin flip process, the Glauber part. This process has a Reaction-Diffusion equation as hydrodynamic limit, as is proven by De Masi et.al.. The ergodicity of these dynamics (one well potential) was proven in Brassesco el.al., for any dimension. In this article we prove the asymptotic exponentiality for certain exit time from a subset of the basin of attraction of the well.

Keywords: exit times, interacting particle systems, Glauber-Kawasaki dynamics, reaction-diffusion equations, hydrodynamic limits

AMS Subject Classifications: 60K35 (82C22, 82C31)

1 Introduction

The class of interacting particle systems that we study, was proposed by De Masi, Ferrari and Lebowitz, as alternative models for reaction-diffusion systems. On the configuration space $\{-1,+1\}^{\mathbb{Z}^d}$, they are obtained from the superposition of a Glauber (spin flip) type dynamics, corresponding to the reactive part, and a stirring one, also called Kawasaki dynamics at infinite temperature, that corresponds to the diffusive part, which is speeded by a factor, say ϵ^{-2}. The kinetic limit, which here corresponds to the hydrodynamical one, involves the simultaneous change of space scaling in the diffusive limit for the stirring, to provide the macro scale (macro= ϵ micro). Under such limit, the macroscopic description is verified, being given by a reaction diffusion equation of the form

$$\partial_t m = \Delta m + F(m). \tag{1}$$

where $m(r,t) \in \mathbb{R}$ represents the magnetization or density. The force term $F(\cdot)$ being determined by the Glauber rates. For example, given any polynomial $F(\cdot)$ we may choose finite range spin flip rates which lead to the above equation (of course there are innumerous choices, though the behaviour should be similar). That is, the empirical magnetization, or the density of particles in small boxes, converges in probability to the solution of the reaction diffusion equation. Indeed one knows more: strong forms of propagation of chaos (i.e. asymptotic independence of different spins) have been proven, initially by De Masi et al. [DFL], and later in sharper forms by several authors, see A.De Masi, E.Presutti [DP]. At the level of propagation of chaos, more general systems, with $m(r,t)$ taking values in $\mathbb{R}^n$ are treated just in the same way, though we restrict ourselves to the case $n = 1$. So there is no loss in assuming $F(m) = -V'(m)$ in the above equation.

For a class of rates leading to a single well potential, the ergodicity for any $d \geq 1$, has been proven by Brassesco et. al. [BPSV1]. The same authors also proved ergodicity in the case of a double well potential, provided $V(\cdot)$ has a unique minimizer, cf.Brassesco et. al. [BPSV2]. We also refer to the article of Durrett and Newhauser [DN], where a large classes of spin flip rates were considered, with the characteristic of having

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a trapping configuration (e.g. the contact process). In such case phase transition might occur, for the same reason which leads to the existence of an invariant measure with magnetization different from the trapping configuration, i.e. is related to the minimizer of the potential.

The problem which we want to address refers to the large desviation behaviour of the process, fixing Glauber rates which lead to a one well potential. Our goal is to investigate the asymptotic loss of memory for the exit time from attracting sets contained in the "basin of attraction" of the ergodic measure. Connected to the reaction-diffusion model, related questions have been considered when the system is obtained by the addition of a small random noise to the equation (1), as treated by Martinelli, Olivieri and Scoppola [MOS]; Brassesco, Olivieri, and Vares [BOV], using coupling arguments, together with estimates of large deviations, which in this infinite dimensional setup were obtained by Faris and Jona Lasinio [F], extending Freidlin and Wentzell results. There are many analogous results in the context of low temperature Glauber dynamics on finite volumes, indeed extended to a large class of Markov chains, but they involve rather different methods.

We consider a macroscopically finite volume, i.e. a torus of side ϵ^{-1}, though the result extends to larger volumes, which are tending to infinity in the macro scale, as for instance, ϵ^{-l}, $l > 1$. Also, for simplicity we prove for dimension $d = 1$, and study the problem when we start the process from the Bernoulli measure, which has the mean in one of the basins of the atraction. There is no problem to generalizing to dimensions $d \geq 1$, and to consider the initial configuration starting from an smooth profile, which is contained also in a basin of atraction. We prove that the properly rescaled exit time follows an exponential law. For the proof we use that its possible to couple two evolutions of the process, starting in the same basin of atraction in a time of order $a|\log\epsilon|$, see Brassesco et. al. [BPSV1]. Finally we use a result from Azlarov, Volodin [A], which bound the uniform distance between e^{-t} and $\mathbb{P}(\xi > t)$, (ξ is a random variable) from a bound over $\mathbb{P}(\xi > t + s|\xi > s) - \mathbb{P}(\xi > t)$; this is the loss of memory property.

2 Definitions and results

We consider, for each $\epsilon > 0$, a family of Markov process $\{\sigma_t\}_{t\geq 0}$, $\sigma_t \in \Omega^\epsilon = \{-1, 1\}^{\mathbb{Z}_\epsilon}$, where we let $\mathbb{Z}_\epsilon$ be the set of integers mod$[\epsilon^{-1}]$, $[x]$ is the integer part of $x \in \mathbb{R}$: $\mathbb{Z}_\epsilon = \mathbb{Z}_{[\epsilon^{-1}]} = \{0, 1, \ldots, [\epsilon^{-1}]\}$, with both ends identified. The evolution of the process is governed by the generator:

$$L_\mu^{(\epsilon)} = \epsilon^{-2} L_0 + L_\mu, \tag{2}$$

where L_0 is the generator of the stirring process:

$$L_0 f(\sigma) = \sum_{|x-y|=1} [f(\sigma^{x,y}) - f(\sigma)], \tag{3}$$

f is a function in Ω^ϵ, $x, y \in \mathbb{Z}_\epsilon$, and

$$\sigma^{x,y}(z) = \begin{cases} \sigma(z) & z \neq x, y; \\ \sigma(y) & z = x; \\ \sigma(x) & z = y, \end{cases}$$

and L_μ is the generator of the spin flip process:

$$L_\mu f(\sigma) = \sum_{x\in\mathbb{Z}_\epsilon} c_\mu(x, \sigma)[f(\sigma^x) - f(\sigma)], \tag{4}$$

again f is a function in Ω^ϵ, and

$$\sigma^x(z) = \begin{cases} -\sigma(z) & z = x; \\ \sigma(z) & z \neq x. \end{cases}$$

$c_\mu(x, \sigma)$, the spin flip intensity, is:

$$c_\mu(x, \sigma) = c_0(x, \sigma) - \frac{\mu}{2}\sigma(x), \tag{5}$$

and $c_0(x,\sigma)$ is defined as:

$$c_0(x,\sigma) = 1 - \gamma\sigma(x)[\sigma(x-1) + \sigma(x+1)] + \gamma^2\sigma(x-1)\sigma(x+1), \tag{6}$$

μ is restricted to $0 \leq \mu < 2(1-\gamma)^2$ for $c_\mu(x,\sigma)$ to be positive, we will consider in this article the case of $\mu = 0$,and $\gamma \in (0, \frac{1}{2})$, so $c_\mu(x,\sigma) = c_0(x,\sigma)$. The process defined above is called the Glauber+Kawasaki process.

If ν is a probability on Ω^ϵ (resp. a single configuration σ), we will denote by $E_\nu^{(\epsilon)}$, (resp. by $E_\sigma^{(\epsilon)}$) the expectation of the above process starting with law ν (resp. from the configuration σ); also we write $P_\nu^{(\epsilon)}$ (resp. $P_\sigma^{(\epsilon)}$) when we refer to the law of the process. In [DFL] it was proved that, as $\epsilon \to 0$, the process, in infinite volume, converges to the solution of the reaction-diffusion equation:

$$\frac{\partial m}{\partial t} = \Delta m - V'_\mu(m), \tag{7}$$

where :

$$-V'_\mu = \mathbb{E}_{\nu_m}(-2\sigma(0)\, c_\mu(0,\sigma)) = -\alpha m - \beta m^3 + \mu, \tag{8}$$

ν_m denoting the Bernoulli product measure on Ω^ϵ with average m and

$$\alpha = 2(1-2\gamma), \beta = 2\gamma^2. \tag{9}$$

In the case $\mu = 0$, the derivative of the potential is:

$$-V' = \mathbb{E}_{\nu_m}(-2\sigma(0)\, c(0,\sigma)) = -\alpha m - \beta m^3$$

The polynomial V'_μ is the derivative of the, in general double well, potential V_μ (see fig.1), and in the case we will consider ($\mu = 0$,and $\gamma \in (0, \frac{1}{2})$) that potential V has only one well, with a minimum at 0.

The case of infinite volume (ie. $\{-1,1\}^{\mathbb{Z}^d}$) was considered in [BPSV1] (one well) and in [BPSV2] (two wells), with the same dynamics. It was proved there that for ϵ small enough the process is ergodic (in our case the process is immediately ergodic, since we are in finite volume). Observe that (7) is not ergodic (in the sense that it has two invariant solutions), also that $m = 0$ is the minimum in the case that we consider.

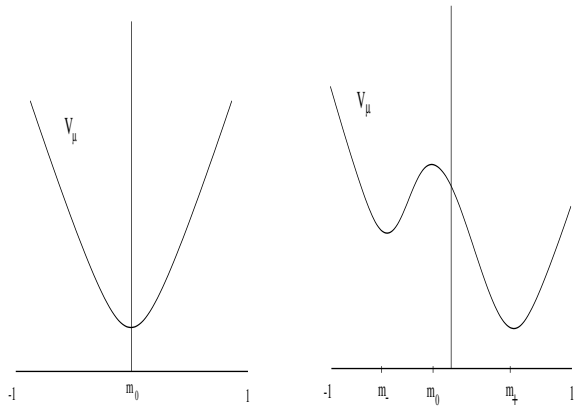

Figure 1

We shall consider the set of configurations with empirical magnetization in small intervals (of size ϵ^{-a}, where $0 < a < 1$ is close to one), belongs to $[m_- - \delta, m_- + \delta]$, with δ small enough, i.e.

$$A_\delta^{(\epsilon)} = \{\sigma \in \Omega^\epsilon : |\epsilon^a \sum_{x:|x| \leq \frac{\epsilon^{-a}}{2}} \sigma(x+y)| < \delta, \quad \forall y \in \mathbb{Z}_\epsilon\}. \tag{10}$$

When it is clear from the context we drop the superscript ϵ and write simply A_δ. Now consider the Bernoulli product measure in Ω^ϵ with average $m = 0$, ν_m, and define the following stopping time

$$\tau^\epsilon = \inf\left\{t > 0 : \sigma_t \notin A_\delta^{(\epsilon)}\right\}.$$

Also, define β_ϵ as: $P_{\nu_m}^{(\epsilon)}(\tau^\epsilon > \beta_\epsilon) = e^{-1}$. Regarding the asymptotic distribution of $\tau^\epsilon/\beta_\epsilon$, the main result is:

Theorem 2.1 For τ^ϵ defined above it holds:

$$\lim_{\epsilon\to 0}\sup_{t\geq 0}|P_{\nu_m}^{(\epsilon)}(\tau^\epsilon > t\beta_\epsilon) - e^{-t}| = 0. \tag{11}$$

3 Proof of Theorem 2.1

We shall prove that if we start the process from the Bernoulli product measure, ν_{m_-}, then, under $P^{(\epsilon)}$, $\tau^\epsilon/\beta_\epsilon$ converges, in the Levy metric, to a mean one exponential r.v.. First we prove that $\tau^\epsilon/\beta_\epsilon$ is uniformly close, in the Levy metric, to a family of exponentials r.v.; and then that this family converges also in the Levy metric to a mean one exponential r.v.. As convergence in Levy metric, with an absolutely continuous limit, implies convergence in Kolmogorov metric, then the theorem is proved.

Let $k \geq 1$ be fixed. In Hinojosa [H] it is proved the following lower bound for β_ϵ: for any $n \geq 1$, there exists $c_n > 0$, such that for $\epsilon > 0$ small enough it holds:

$$P_{\nu_m}^{(\epsilon)}\left(\tau^\epsilon < \epsilon^{-k}\right) \leq c_n\epsilon^n. \tag{12}$$

Note that, by the Cramer-Chernov theorem for large deviations for i.i.d. (Bernoulli) random variables, for any $\delta' > 0$, at time zero

$$P_{\nu_m}^{(\epsilon)}(A_{\delta'}^c) \leq \epsilon^{-1}P_{\nu_m}^{(\epsilon)}\left(|\epsilon^a \sum_{|x|\leq\frac{\epsilon^{-a}}{2}} \sigma(x)| \geq \delta'\right) \leq \epsilon^{-1}e^{-\tilde{c}\epsilon^{-a}} \leq e^{-c\epsilon^{-a}}, \tag{13}$$

For some $c > 0$. Then we shall consider that with large ν_m-probability the initial configuration starts from $A_{\delta'}$, for any $\delta' < \delta$. Also, since (12), if ϵ is small enough we have that:

$$P_{\nu_m}^{(\epsilon)}\left(\tau^\epsilon > \epsilon^{-k}\right) > e^{-1}.$$

But by definition $P_{\nu_m}^{(\epsilon)}(\tau^\epsilon > \beta_\epsilon) = e^{-1}$, therefore $\epsilon^{-k} < \beta_\epsilon$, for any $k \geq 1$, and ϵ small enough.

The main part of the proof is contained in Proposition 3.1. It says that the family of r.v. $\tau^\epsilon/\beta_\epsilon$ has, uniformly in ϵ, loss of memory property (note that this feature caracterizes the exponential distribution). From this Proposition, we shall prove that, in fact, for each ϵ we have an exponential random variable that is close, in the Levy metric sense, to $\tau^\epsilon/\beta_\epsilon$.

Before stating the loss of memory property, we let $\gamma_\epsilon = \tilde{a}|\log\epsilon|$, where $\tilde{a}$ is a positive constant that will be fixed later. Observe that

$$\frac{\gamma_\epsilon}{\beta_\epsilon} \to 0 \text{ as } \epsilon \to 0,$$

and, since (12)

$$P_{\nu_m}^{(\epsilon)}(\tau^\epsilon < \gamma_\epsilon) \leq c_n\epsilon^n. \tag{14}$$

Also let $F_\epsilon(t) = P_{\nu_m}^{(\epsilon)}(\tau^\epsilon > t\beta_\epsilon)$ and

$$h_\epsilon(t,s) = \frac{F_\epsilon(t+s)}{F_\epsilon(s)} - F_\epsilon(t).$$

Proposition 3.1 For any $n \geq 0$, there exists c_n such that for ϵ small enough it holds

$$\sup_{s\geq 0, t\geq \frac{\gamma_\epsilon}{\beta_\epsilon}} |h_\epsilon(t,s)| \leq c_n \epsilon^n.$$

We shall prove this proposition in the next section.

This result will allow us to prove

Proposition 3.2 For any $n \geq 1$ there exists c_n such that for any ϵ small enough the following holds

$$d_L\left(\frac{\tau^\epsilon}{\beta_\epsilon}, \mathcal{E}(\lambda_\epsilon)\right) \leq c_n \epsilon^n,$$

where $d_L(\cdot,\cdot)$ refers to the Levy metric between two random variables, i.e.

$$d_L(X,Y) = \inf\{\varrho > 0 : P(X > t+\varrho) - \varrho \leq P(y > t) \leq P(X > t-\varrho) + \varrho, \forall t \in \mathbb{R}\},$$

and $\mathcal{E}(\lambda)$ is a exponential random variable with mean λ^{-1}, recall that $\lambda_\epsilon^{-1} = \mathbb{E}^{(\epsilon)}_{\nu_{m_-}} \frac{\tau^\epsilon}{\beta_\epsilon}$. We defer the proof of this Proposition to the end of this section.

Finally the Theorem 2.1 will be proved after showing that $\lambda_\epsilon \to 1$, since this implies that $d_L\left(\mathcal{E}(1), \mathcal{E}(\lambda_\epsilon)\right) \to 0$, and then

$$d_L\left(\frac{\tau^\epsilon}{\beta_\epsilon}, \mathcal{E}(1)\right) \to 0.$$

The proof that $\lambda_\epsilon \to 1$ follows easily. Indeed,following Aslarov,[A] we have that $\frac{\tau^\epsilon}{\beta_\epsilon}$ has finite moments of all orders, in particular $\lambda_\epsilon^{-1} = \mathbb{E}^{(\epsilon)}_{\nu_m} \frac{\tau^\epsilon}{\beta_\epsilon} < \infty$.

By definition $P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > \beta_\epsilon) = e^{-1}$, and this implies that $e^{-1} \leq \lambda_\epsilon^{-1}$, so that $0 < \lambda_\epsilon \leq e$, for all $\epsilon > 0$. Take any $\varrho \geq 0$ in the the defining property of $d_L(\frac{\tau^\epsilon}{\beta_\epsilon}, \mathcal{E}(\lambda_\epsilon))$ and $t = 1$, so that :

$$e^{-\lambda_\epsilon(1+\varrho)} - \varrho \leq e^{-1} \leq e^{-\lambda_\epsilon(1-\varrho)} + \varrho,$$

then

$$\begin{aligned} e^{-1} - e^{-\lambda_\epsilon} &\leq e^{-\lambda_\epsilon}(e^{\lambda_\epsilon \varrho} - 1) + \varrho \\ &\leq \varrho(e^{e\varrho+1} + 1), \end{aligned}$$

also

$$e^{-1} - e^{-\lambda_\epsilon} \geq \left(e^{-e\varrho} - 1\right) - \varrho \geq -\varrho(e^{1-e\varrho} + 1), \tag{15}$$

but the infimum of such ϱ is less than $c_n\epsilon^n$, so that

$$|e^{-1} - e^{-\lambda_\epsilon}| \leq c_n' \epsilon^n,$$

and then $\lambda_\epsilon \to 1$, as ϵ goes to zero.■

Proof of Proposition 3.2

It's enough to prove that the two inequalities in the definition of $d_L(\frac{\tau^\epsilon}{\beta_\epsilon}, \mathcal{E}(\lambda_\epsilon))$ hold for $\varrho = c_n\epsilon^n$, with $n \geq 1$.

Take $\theta_\epsilon(t) = \lambda_\epsilon \int_0^\infty P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon)\, h_\epsilon(t,s)ds$, note that $|\theta_\epsilon(t)| \leq 1$, for all t, and by Proposition 3.1, for $t \geq \frac{\gamma_\epsilon}{\beta_\epsilon}$, $|\theta_\epsilon(t)| \leq c_n\epsilon^n$. From Aslarov [A], the following representation of $F_\epsilon(t)$ holds :

$$F_\epsilon(t) = e^{-\lambda_\epsilon t} + \lambda_\epsilon \int_0^t e^{-\lambda_\epsilon(t-u)}\theta_\epsilon(u)du - \theta_\epsilon(t).$$

We begin with the first inequality of the Levy metric. Since, for $k > n$ large and ϵ small enough $\frac{\gamma_\epsilon}{2c_n\epsilon^n} \leq \epsilon^{-k} < \beta_\epsilon$, then $\frac{\gamma_\epsilon}{\beta_\epsilon} \ll 2c_n\epsilon^n = \varrho$, So that $t+\varrho > \frac{\gamma_\epsilon}{\beta_\epsilon}$, and we can use the Proposition 3.1 to get

$$\begin{aligned} F_\epsilon(t+\varrho) &= e^{-\lambda_\epsilon(t+\varrho)} + \lambda_\epsilon \int_0^{(t+\varrho)} e^{-\lambda_\epsilon(t+\varrho-u)}\theta_\epsilon(u)du - \theta_\epsilon(t+\varrho) \\ &\leq e^{-\lambda_\epsilon(t+\varrho)} + \lambda_\epsilon \int_0^{\frac{\gamma_\epsilon}{\beta_\epsilon}} e^{-\lambda_\epsilon(t+\varrho-u)}du + c_n\epsilon^n\lambda_\epsilon \int_{\frac{\gamma_\epsilon}{\beta_\epsilon}}^{(t+\varrho)} e^{-\lambda_\epsilon(t+\varrho-u)}du + c_n\epsilon^n \\ &\leq e^{-\lambda_\epsilon t}e^{-\lambda_\epsilon(\varrho-\frac{\gamma_\epsilon}{\beta_\epsilon})} + \varrho \leq e^{-\lambda_\epsilon t} + \varrho. \end{aligned} \tag{16}$$

Now for the other inequality in the Levy metric, we must analise various cases

i) If $t < \varrho$, then $F_\epsilon(t-\varrho) = 1$ so obviously $F_\epsilon(t-\varrho) \geq e^{-\lambda_\epsilon t} - \varrho$.

ii) If $t > \varrho$ and $t - \varrho < \frac{\gamma_\epsilon}{\beta_\epsilon}$ then $F_\epsilon(t-\varrho) \geq F_\epsilon(\frac{\gamma_\epsilon}{\beta_\epsilon}) \geq 1 - c_n\epsilon^n \geq e^{-\lambda_\epsilon t} - \varrho$, by Eqn.(14).

iii)If $t > \varrho$ and $t - \varrho > \frac{\gamma_\epsilon}{\beta_\epsilon}$ then

$$\begin{aligned} F_\epsilon(t-\varrho) &= e^{-\lambda_\epsilon(t-\varrho)} + \lambda_\epsilon \int_0^{(t-\varrho)} e^{-\lambda_\epsilon(t-\varrho-u)}\theta_\epsilon(u)du - \theta_\epsilon(t-\varrho) \\ &\geq e^{-\lambda_\epsilon(t-\varrho)} - \lambda_\epsilon \int_0^{\frac{\gamma_\epsilon}{\beta_\epsilon}} e^{-\lambda_\epsilon(t-\varrho-u)}du - c_n\epsilon^n\lambda_\epsilon \int_{\frac{\gamma_\epsilon}{\beta_\epsilon}}^{(t+\varrho)} e^{-\lambda_\epsilon(t+\varrho-u)}du - c_n\epsilon^n \\ &\geq e^{-\lambda_\epsilon(t-\varrho)} - (e^{-\lambda_\epsilon(t-\varrho-\frac{\gamma_\epsilon}{\beta_\epsilon})} - e^{-\lambda_\epsilon(t-\varrho)}) - \varrho \\ &= e^{-\lambda_\epsilon t}[2e^{\lambda_\epsilon\varrho} - e^{\lambda_\epsilon(\varrho+\frac{\gamma_\epsilon}{\beta_\epsilon})}] - \varrho. \end{aligned} \tag{17}$$

All we need to prove is that $2e^{\lambda_\epsilon\varrho} - e^{\lambda_\epsilon(\varrho+\frac{\gamma_\epsilon}{\beta_\epsilon})} > 1$. To this end, first observe that for z close to 1 we have that $2z - 1 > z^{3/2}$. Since $\varrho = 2c_n\epsilon^n$ and $0 \leq \lambda_\epsilon \leq e$ we have that $\lambda_\epsilon\varrho \to 0$ as $\epsilon \to 0$, so that taking $z = e^{\lambda_\epsilon\varrho}$, for ϵ small enough,

$$2e^{\lambda_\epsilon\varrho} - e^{3/2\lambda_\epsilon\varrho} > 1.$$

But $\frac{\gamma_\epsilon}{\beta_\epsilon} < \varrho/2$, so $e^{\lambda_\epsilon(\varrho+\frac{\gamma_\epsilon}{\beta_\epsilon})} < e^{3/2\lambda_\epsilon\varrho}$, using this in (17) we can conclude that

$$F_\epsilon(t-\varrho) \geq e^{-\lambda_\epsilon t} - \varrho.$$

So, the infimum in the Levy distance, is less than $\varrho = c_n\epsilon^n$, and the Proposition is proved.■

4 Loss of Memory Property

4.1 Loss of Memory Property

Proof of Proposition 3.1

We shall prove that

$$|P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > (t+s)\beta_\epsilon) - P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > t\beta_\epsilon)P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon)| \leq c_n\epsilon^n P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon), \tag{18}$$

for $t \geq \frac{\gamma_\epsilon}{\beta_\epsilon}$. To do this, we shall condition on the first sumand at time $s\beta_\epsilon + \gamma_\epsilon$, and after some algebra we will get the difference between terms like $P^{(\epsilon)}_\sigma(\tau^\epsilon > t\beta_\epsilon)$, $\sigma \in A^{(\epsilon)}_{\delta'}$, multiplied by $P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon)$, with $0 < \delta' < \delta$. So, we need to bound such differences. The main tool is the following Proposition, that permits us to couple two processes before the exit time of $A^{(\epsilon)}_\delta$. Recall that $\gamma_\epsilon = \tilde{a}|\log\epsilon|$.

Proposition 4.1 Let $0 < \delta' < \delta$. For any $n \geq 0$ there exists $c_n > 0$ such that for any $\epsilon > 0$ small enough, the following holds:

$$\sup_{t\geq 0} |P^{(\epsilon)}_{\sigma}(\tau^\epsilon > t\beta_\epsilon) - P^{(\epsilon)}_{\sigma'}(\tau^\epsilon > t\beta_\epsilon)| \leq P^{(\epsilon)}_{\sigma}(\tau^\epsilon < \gamma_\epsilon) + P^{(\epsilon)}_{\sigma'}(\tau^\epsilon < \gamma_\epsilon) + c_n\epsilon^n, \tag{19}$$

for any $\sigma, \sigma' \in A^{(\epsilon)}_{\delta'}$.

We defer the proof to the end of the section. Turning to the proof of (18), note that $\gamma_\epsilon < \beta_\epsilon$ and for $n \geq 1$ there exists $c_n > 0$ such that

$$P^{(\epsilon)}_{\nu_m}(\tau^\epsilon < \gamma_\epsilon) \leq c_n\epsilon^n,$$

for ϵ small enough. We begin by bounding from above

$$P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > (t+s)\beta_\epsilon) - P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > t\beta_\epsilon)P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon).$$

Since, by hypothesis, $t\beta_\epsilon \geq \gamma_\epsilon$, we can condition the process up to time $s\beta_\epsilon + \gamma_\epsilon$ and after using the Markov property, we get that this expression is less than

$$\begin{aligned}
&\mathbb{E}^{(\epsilon)}_{\nu_m}(\mathbf{1}_{\tau^\epsilon > s\beta_\epsilon} P^{(\epsilon)}_{\sigma_{s\beta_\epsilon+\gamma_\epsilon}}(\tau^\epsilon > t\beta_\epsilon - \gamma_\epsilon)) - P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > t\beta_\epsilon - \gamma_\epsilon)P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon)\\
&\quad + \quad P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > t\beta_\epsilon - \gamma_\epsilon)P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon) - P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > t\beta_\epsilon)P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon).
\end{aligned}$$

Decomposing the first summand according to $\{\sigma_{s\beta_\epsilon+\gamma_\epsilon} \in A_{\delta'}\}$, or not, (recall that $\delta' < \delta$) the last expression is

$$\begin{aligned}
&\mathbb{E}^{(\epsilon)}_{\nu_m}(\mathbf{1}_{\tau^\epsilon > s\beta_\epsilon}\mathbf{1}_{\sigma_{s\beta_\epsilon+\gamma_\epsilon}\in A_{\delta'}}[P^{(\epsilon)}_{\sigma_{s\beta_\epsilon+\gamma_\epsilon}}(\tau^\epsilon > t\beta_\epsilon - \gamma_\epsilon) - P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > t\beta_\epsilon - \gamma_\epsilon)])\\
&\quad + \mathbb{E}^{(\epsilon)}_{\nu_m}(\mathbf{1}_{\tau^\epsilon > s\beta_\epsilon}\mathbf{1}_{\sigma_{s\beta_\epsilon+\gamma_\epsilon}\notin A_{\delta'}}[P^{(\epsilon)}_{\sigma_{s\beta_\epsilon+\gamma_\epsilon}}(\tau^\epsilon > t\beta_\epsilon - \gamma_\epsilon) - P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > t\beta_\epsilon - \gamma_\epsilon)])\\
&\quad + P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon)[P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > t\beta_\epsilon - \gamma_\epsilon) - P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > t\beta_\epsilon)]).
\end{aligned} \tag{20}$$

By Proposition 4.1, the first term in (20) can be bounded by:

$$\mathbb{E}^{(\epsilon)}_{\nu_m}(\mathbf{1}_{\tau^\epsilon > s\beta_\epsilon}\mathbf{1}_{\sigma_{s\beta_\epsilon+\gamma_\epsilon}\in A_{\delta'}}P^{(\epsilon)}_{\sigma_{s\beta_\epsilon+\gamma_\epsilon}}(\tau^\epsilon \leq \gamma_\epsilon)) + c_n\epsilon^n P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon).$$

The first sumand could be expressed, after conditioning up to $s\beta_\epsilon + \frac{\gamma_\epsilon}{2}$, as

$$\begin{aligned}
\mathbb{E}^{(\epsilon)}_{\nu_m}\left(\mathbf{1}_{\tau^\epsilon > s\beta_\epsilon}\mathbf{1}_{\sigma_{s\beta_\epsilon+\frac{\gamma_\epsilon}{2}}\in A_{\delta'}}P^{(\epsilon)}_{\sigma_{s\beta_\epsilon+\frac{\gamma_\epsilon}{2}}}(\exists u, \frac{\gamma_\epsilon}{2} \leq u \leq \frac{3\gamma_\epsilon}{2} \quad \sigma_u \notin A_\delta)\right)\\
+c_n\epsilon^n P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon).
\end{aligned}$$

The conditional probability that appears inside, could be bounded by $c_n\epsilon^n$, since if $\sigma \in A_\delta$ then, at times of order γ_ϵ, it belongs to $A_{\delta'}, \delta > \delta'$. Then we divide the time interval $[\frac{\gamma_\epsilon}{2}, \frac{3\gamma_\epsilon}{2}]$ in subintervals of size $\epsilon^\rho, \rho > 0$, in each of them, also with large probability, the process stays in A_δ. (We skip the details, see Hinojosa [H]). So that the first sumand in (20) can be bounded by

$$P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon)c_n\epsilon^n.$$

In similar way for the second term of (20), after conditiong up to time $s\beta_\epsilon$:

$$\begin{aligned}
&2\mathbb{E}^{(\epsilon)}_{\nu_m}(\mathbf{1}_{\tau^\epsilon > s\beta_\epsilon}\mathbf{1}_{\sigma_{s\beta_\epsilon+\gamma_\epsilon}\notin A_{\delta'}})\\
&\quad \leq \quad 2\mathbb{E}^{(\epsilon)}_{\nu_m}(\mathbf{1}_{\tau^\epsilon > s\beta_\epsilon}P^{(\epsilon)}_{\sigma_{s\beta_\epsilon}}(\sigma_{\gamma_\epsilon} \notin A_{\delta'}))\\
&\quad \leq \quad 2P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon)c_n\epsilon^n,
\end{aligned} \tag{21}$$

where we used again Hinojosa [H] , in the second inequality.

Finally, the third term in (20), after decomposing according to the event $\{\sigma_{\gamma_\epsilon} \in A_{\delta'}\}$, and using the Proposition 4.1. As above, we get the same bound:

$$\begin{aligned} & P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon)\left(P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > t\beta_\epsilon - \gamma_\epsilon) - \mathbb{E}^{(\epsilon)}_{\nu_m}(\mathbf{1}_{\tau^\epsilon > \gamma_\epsilon} P^{(\epsilon)}_{\sigma_{\gamma_\epsilon}}(\tau^\epsilon > t\beta_\epsilon - \gamma_\epsilon))\right) \\ \leq \; & P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon)\left(2P^{(\epsilon)}_{\nu_m}(\tau^\epsilon \leq \gamma_\epsilon) + c_n\epsilon^n + \mathbb{E}^{(\epsilon)}_{\nu_m}(\mathbf{1}_{\sigma_{\gamma_\epsilon}\in A_{\delta'}} P^{(\epsilon)}_{\sigma_{\gamma_\epsilon}}(\tau^\epsilon \leq \gamma_\epsilon))\right) \\ \leq \; & P^{(\epsilon)}_{\nu_{m_-}}(\tau^\epsilon > s\beta_\epsilon)c_n\epsilon^n. \end{aligned}$$

So, until now we proved that

$$P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > (t+s)\beta_\epsilon) - P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > t\beta_\epsilon)P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon). \tag{22}$$

is bounded from above by $P^{(\epsilon)}_{\nu_m}(\tau^\epsilon > s\beta_\epsilon)c_n\epsilon^n$, for $s > 0$ and $t > \frac{\gamma_\epsilon}{\beta_\epsilon}$. Now, proceeding similarly we shall get the same bound by below, and the proposition is proved.■

Proof of Proposition 4.1

Note that, if we take $\sigma, \sigma' \in A^{(\epsilon)}_{\delta'}$, then

$$\begin{aligned} & \left|P^{(\epsilon)}_{\sigma}(\tau^\epsilon_\sigma > t\beta_\epsilon) - P^{(\epsilon)}_{\sigma'}(\tau^\epsilon_{\sigma'} > t\beta_\epsilon)\right| \\ \leq \; & P^{(\epsilon)}_{\sigma,\sigma'}(\tau^\epsilon_\sigma > t\beta_\epsilon, \tau^\epsilon_{\sigma'} \leq t\beta_\epsilon) + P^{(\epsilon)}_{\sigma,\sigma'}(\tau^\epsilon_{\sigma'} > t\beta_\epsilon, \tau^\epsilon_\sigma \leq t\beta_\epsilon) \\ \leq \; & P^{(\epsilon)}_{\sigma,\sigma'}(\tau^\epsilon_\sigma > t\beta_\epsilon, \tau^\epsilon_{\sigma'} < \gamma_\epsilon) + P^{(\epsilon)}_{\sigma,\sigma'}(\tau^\epsilon_\sigma > t\beta_\epsilon, \gamma_\epsilon \leq \tau^\epsilon_{\sigma'} \leq t\beta_\epsilon) \\ + \; & P^{(\epsilon)}_{\sigma,\sigma'}(\tau^\epsilon_{\sigma'} > t\beta_\epsilon, \tau^\epsilon_\sigma < \gamma_\epsilon) + P^{(\epsilon)}_{\sigma,\sigma'}(\tau^\epsilon_{\sigma'} > t\beta_\epsilon, \gamma_\epsilon \leq \tau^\epsilon_\sigma \leq t\beta_\epsilon) \\ \leq \; & P^{(\epsilon)}_{\sigma}(\tau^\epsilon < \gamma_\epsilon) + P^{(\epsilon)}_{\sigma'}(\tau^\epsilon < \gamma_\epsilon) + 2P^{(\epsilon)}_{\sigma,\sigma'}(\sigma_{\gamma_\epsilon} \neq {\sigma'}_{\gamma_\epsilon}). \end{aligned} \tag{23}$$

For the last term, we have

$$\begin{aligned} P^{(\epsilon)}_{\sigma,\sigma'}\left(\sigma_{\gamma_\epsilon} \neq \sigma'_{\gamma_\epsilon}\right) \; & \leq \; \sum_{x\in\mathbb{Z}_\epsilon} P^{(\epsilon)}_{\sigma,\sigma'}\left(\sigma_{\gamma_\epsilon}(x) \neq \sigma'_{\gamma_\epsilon}(x)\right) \\ & \leq \; \epsilon^{-1}\mathbb{E}^{(\epsilon)}_{\sigma,\sigma'}\left|\sigma_{\gamma_\epsilon}(0) - \sigma'_{\gamma_\epsilon}(0)\right|. \end{aligned} \tag{24}$$

in the last inequality by translation invariance.

Thus, the main step of the proof is to get a bound for the last term. A bound for the discrepancies between σ^-_t and σ^+_t, appears in [BPSV1], when the process starts from $-\mathbf{1}$ and $\mathbf{1}$ (in infinite volume):

Proposition 4.2 : For any $\gamma < \frac{1}{2}$, $n \geq 1$ and $\epsilon > 0$ small enough there exists positive constants $\tilde{a}$ and c_n such that :

$$0 \leq \mathbb{E}^{(\epsilon)}_{\sigma,\sigma'}\left|\sigma_T(0) - \sigma'_T(0)\right| \leq c_n\epsilon^n,$$

where $T := \tilde{a}\left|\ln\epsilon\right|$, and $\sigma, \sigma' \in A_\delta$.

Using this result , the right hand side of (24) could be bounded by $c_n\epsilon^{n-1}$ uniformly in A_δ, and the Proposition 4.1 is proved.■

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