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Abstract

We describe a methodology suitable for analysis of generalized $M/G/C/C$ state dependent queueing systems and make available a useful software tool. Programmed in an object-oriented way, the computer codes form a basic building block useful in many applications and contexts. The programs compute important performance measurements, such as the blocking probabilities, the number of customers in the system, throughput, and waiting time. Computational results are provided.

Keywords: Queueing systems, finite capacity, state dependent.

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1 Introduction

Waiting in line is one of the most common annoyances modern society has to face. Nobody likes to wait and no manager of the service establishments likes us to wait since it costs them business. A line (or queue) is formed every time there is more demand than service available. In order to answer crucial questions such as "what should the level of service be to ensure satisfactory service?", one needs to know how long a customer may be willing to wait and how many customers are expected to be in the line (Gross and Harris, 1998).

Our objective with this paper is to present a general methodology suitable for analysis of queueing networks as well as to present an object-oriented software, developed in C^{++}, for analysis of $M/G/C/C$ (**M**arkovian interarrival-time distribution, **G**eneral service-time distribution, C parallel servers, and total capacity C) queueing networks, assuming a service time dependent on the current system congestion. The classes are presented in Appendix A.

The importance of this model could be better stated by considering the number of papers that has been published recently on the subject. Among other applications, the state dependent $M/G/C/C$ model is embedded in modern topological network design problems (Mitchell, 1995, MacGregor Smith, 1996), industrial engineering problems (MacGregor Smith and Daskalaki, 1988, MacGregor Smith and Chikhale, 1995), modeling of vehicular traffic flows (MacGregor Smith, 1994, Jain and MacGregor Smith, 1997), and pedestrian flow networks (Yuhaski and MacGregor Smith, 1989, Bakuli and MacGregor Smith, 1996).

This paper is outlined as follows. In Section 2, we present a literature review and the analytical congestion model which is applied to a single queue. The generalized Expansion Method makes possible the analysis of more complex topologies and it is presented in Section 3. In Section 4, we show some computational results that could give some insight into the system behavior. Finally, Section 5 finishes the paper with concluding remarks.

2 Basic Congestion Models

2.1 Motivation

Probably, one of the most interesting problems that could be treated by the methodology presented here is related to pedestrian flows (Tregenza, 1976, Yuhaski and MacGregor Smith, 1989, MacGregor Smith, 1994, Bakuli and MacGregor Smith, 1996). Other applications where the state dependent model can be of use is in vehicular traffic modelling (Jain and MacGregor Smith, 1997), material handling systems (Thumsi and MacGregor Smith, 1998) and other applications where the velocity of customers decrease with incresing customer traffic.

Many studies of pedestrian walking speeds have been conducted. Most of these have been of pedestrian flows at street crossings. Within building interiors, these studies have mainly dealt with fire escape routes. Since the conditions under which differing studies have been conducted may vary considerably, care must be taken when comparing one set

of results with another. However, there are several common factors that can be associated with these different pedestrian walking speed studies. For instance, men tend to walk faster than women do. Mean walking speeds tend to vary among people of different age groups, as adolescents generally walk faster than adults do. A general decline in walking speed is observed with an increase in age. Finally, a trip purpose can largely affect the walking speed of a pedestrian.

As the traffic density increases, however, these factors have a much smaller effect on the walking speed of individual pedestrians. Within a corridor, the progress of faster moving pedestrians tends to become impeded by slower moving pedestrians as the limited floor space of the corridor becomes occupied by more pedestrians. This is because of the decrease in available clear area for passing and avoiding slower moving pedestrians. As the crowd density, defined as the number of pedestrians per unit area, increases, the variability of the walking speeds of individual pedestrians tends to decrease. When the crowd density becomes even slightly moderate, crowd density tends to become the most significant factor in determining pedestrian walking speeds.

Figure 1 presents experimental curves (a through f) that relate the walking speed of a pedestrian to the crowd density, based on various empirical studies (Tregenza, 1976) where it is stated that at a mean density of 3 pedestrians per square meter (ped/m^2), walking is reduced to a shuffle, and at 5 ped/m^2, forward movement is halted. Based on these empirical curves, linear and exponential analytical models for uni-directional walking speeds as a function of the number of pedestrians in a corridor have been developed (Yuhaski and MacGregor Smith, 1989).

2.2   Analysis of Single Queues

The effect we would like to represent is shown in Figure 2 in which the pedestrian through-put versus the arrival rate λ, in a 8×2.5 m corridor, can be seen. The throughput reaches a maximum value and beyond that value start to decrease until it reaches, finally, a flat region. This is an effect, well-know in practice, of the congestion. In order to develop the model, we shall concentrate in the pedestrian flow example, but the methodology could be extended to other contexts.

We are assuming that the time of arrival to a corridor for each pedestrian is independent of that of other pedestrians arriving at the entrance of the corridor. Therefore, we assume that pedestrians arrive to the corridor according to a Poisson process with rate λ. The M in the model represents this exponential arrival to the corridor.

In the $M/G/C/C$ state dependent queueing model of a corridor, the corridor behaves as the server to its occupants. The number of servers is equal to the capacity of the corridor, C. Pedestrians that arrive to the corridor when it is at capacity may not enter the corridor. The queue consists entirely of the corridor without any additional buffer floor space. Therefore, the total number of pedestrians that are allowed in the entire queueing system is also equal to the capacity of the corridor.

The service time of the queueing model of a corridor is equal to the time for a pedestrian to traverse the entire length of the corridor. The rate at which this traversal occurs, the
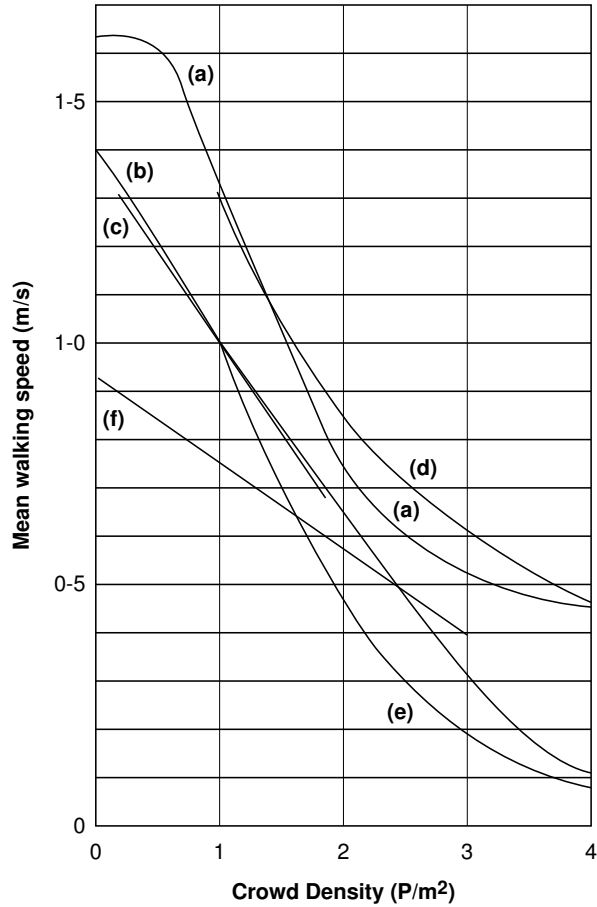
Figure 1: Mean Walking Speed

service rate $f(n)$, is dependent on the number of occupants n within the corridor and follows a general distribution G. We assume that the pedestrian crowd density within the corridor is approximately uniformly distributed throughout the corridor, and that it is this corridor density that determines the average walking speed of a pedestrian within the corridor. The queueing model is state dependent because a change in the number of pedestrians within the corridor will change the service rate of every pedestrian within the corridor. Thus, if there are n pedestrians within a corridor, all of them will have service rate $f(n)$. If there is an arrival to the corridor, and that pedestrian enters the corridor, the service rate will change to $f(n + 1)$. Likewise, if a pedestrian traverses the length of the corridor and leaves the system, the service rate will change to $f(n - 1)$.

The limiting probabilities for the number of pedestrians in a $M/M/C/C$ queueing model have been developed before (Yuhaski and MacGregor Smith, 1989). It has been shown that $M/M/C/C$ and $M/G/C/C$ state dependent queues are stochastically equivalent, as well as that the limiting probabilities for the number of pedestrians in an $M/G/C/C$ state dependent queueing model are as follows (Cheah, 1990):
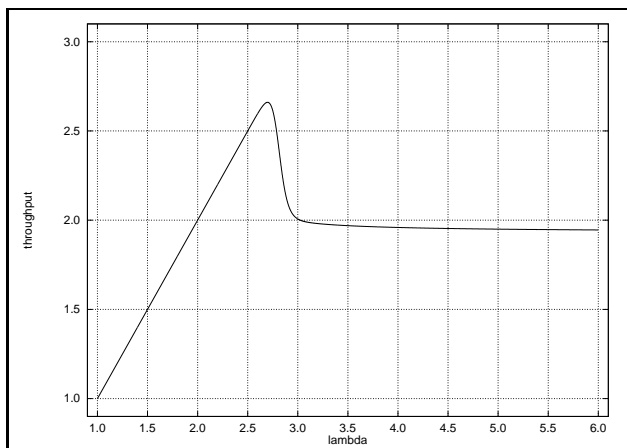
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Figure 2: Throughput Saturation Effect

$$P(n) = \left\{ \frac{[\lambda E(S)]^n}{n! f(n) \cdots f(2) f(1)} \right\} P_0, \tag{1}$$

where

$$P_0^{-1} = 1 + \sum_{i=1}^{C} \left\{ \frac{[\lambda E(S)]^i}{i! f(i) \cdots f(2) f(1)} \right\}, \tag{2}$$

where $n = 1, 2, \ldots, C$, λ is the arrival rate, $E(S) = L/V_1$ is the expected service time of a lone occupant in a corridor of length L considering that $V_1 \approx 1.5$ m/s is the speed of a lone pedestrian, and $f(n) = V_n/V_1$ is the service rate, considered to be the ratio of the average walking speed of n people in the corridor to that of a lone occupant V_1.

The average walking speed of n pedestrians in the corridor could be calculated using several different models. In the following Section, we shall see two of them, the linear and the exponential model for pedestrian walking speeds versus crowd density within the corridor, well suitable for the application (MacGregor Smith, 1994).

2.3   Service Rate Functions

Basically, what one wants is that the congestion model represents the effects depicted in Figure 3, in which the service rate depends on the number of customers in the system.

One possibility is to assume a linear model in which the service rate decays following the expression:

$$f(n) = \frac{V_n}{V_1} = \frac{C + 1 - n}{C}, \tag{3}$$

where V_n is the average walking speed for n pedestrians in a corridor, V_1 the typical walking speed of a lone occupant, and C is the capacity of the corridor which is assumed to be

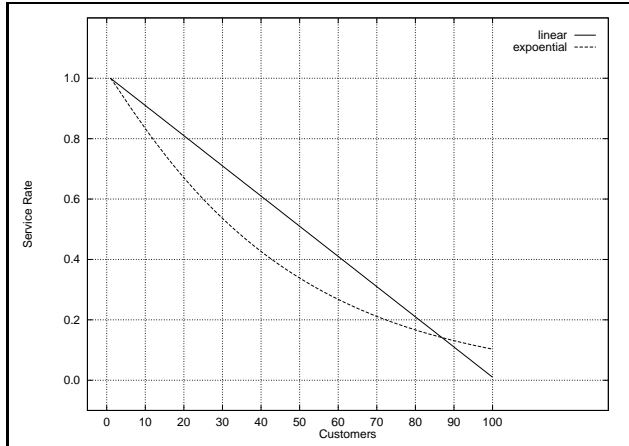$$C = \lfloor kLW \rfloor, \tag{4}$$

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Figure 3: State Dependent Service Rate

where $\lfloor\ \rfloor$ is the floor function, L is the length, W is the width, and k is the capacity of the corridor per square-unit. Considering pedestrian related applications and realizing that k represents the crowd density, a reasonable value for k would be 5, since at this density, movement ceases, in accordance to experimental studies (Tregenza, 1976). The linear model behavior can be seen in Figure 3 drawn for a corridor of 8×2.5 meters.

On the other hand, it is also possible to assume an exponential model in which the service rate decays following the expression:

$$f(n) = \exp\left[-\left(\frac{n-1}{\beta}\right)^{\gamma}\right], \tag{5}$$

where

$$\gamma = \log\left[\frac{\log(V_a/V_1)}{\log(V_b/V_1)}\right] / \log\left(\frac{a-1}{b-1}\right) \tag{6}$$

and

$$\beta = \frac{a-1}{[\log(V_1/V_a)]^{1/\gamma}} = \frac{b-1}{[\log(V_1/V_b)]^{1/\gamma}}, \tag{7}$$

where V_n, V_1, and C are as defined before. The values a and b are arbitrary points used to adjust the curve. In pedestrian related applications, commonly used values are $a = 2LW$ and $b = 4LW$ corresponding to crowd densities of 2 and 4 ped/m². Looking at the curves presented in Figure 1, reasonable values for such points are $V_a = 0.64$ and $V_b = 0.25$. The exponential model behavior is represented in Figure 3, for a corridor of 8×2.5 meters.

As a final remark, we hope it is clear that the methodology is general. Although illustrated here for the case of pedestrians, the model could be adjusted to other applications, as it has been done previously (MacGregor Smith and Daskalaki, 1988, Jain and MacGregor Smith, 1997), or to different data sets.

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3 The Generalized Expansion Method

3.1 Series Queues

The Generalized Expansion Method (Kerbache and MacGregor Smith, 1987) is a combination of the Repeated Trials and Node-by-Node Decomposition approximation methods, with a key characteristic that an artificial holding node is added preceding each finite queue in the network in order to register blocked customer that attempt to enter the finite node when it is at capacity. By the addition of holding nodes, the queueing network is "expanded" into an equivalent Jackson network, in which each node can then be decomposed and analyzed separately.

The Expansion Method consists of three stages that are performed for each node in the original queueing network. To explain these stages, the following notation is established:

h = the holding node established in the Expansion method;

λ = external Poisson arrival rate to the network;

λ_j = Poisson arrival rate to node j;

$\tilde{\lambda}_j$ = effective arrival rate to node j;

μ_j = mean service rate at node j;

$\tilde{\mu}_j$ = effective service rate at node j due to blocking;

P_C = blocking probability of finite queue of size C;

P_C' = feedback blocking probability in the Expansion Method.

Stage 1: Network Reconfiguration

For each node with finite capacity, an artificial node is added directly preceding it, as shown in Figure 4. Customers that attempt to move to the new node but are unable since it is at capacity are re-routed to the artificial node. The probability that an arriving customer is blocked at a finite node equals P_C. Thus, with probability $(1 - P_C)$ it will enter the node, and with probability P_C it will enter the holding node. The holding node is modeled as an $M/G/\infty$ queue, so that there will be no waiting to enter this node.

After service at the holding node, the customer will be blocked with a new probability, P_C'. With probability $(1 - P_C')$, it will proceed to the following node. Otherwise, it must retrace its path through the feedback loop into the artificial node again.
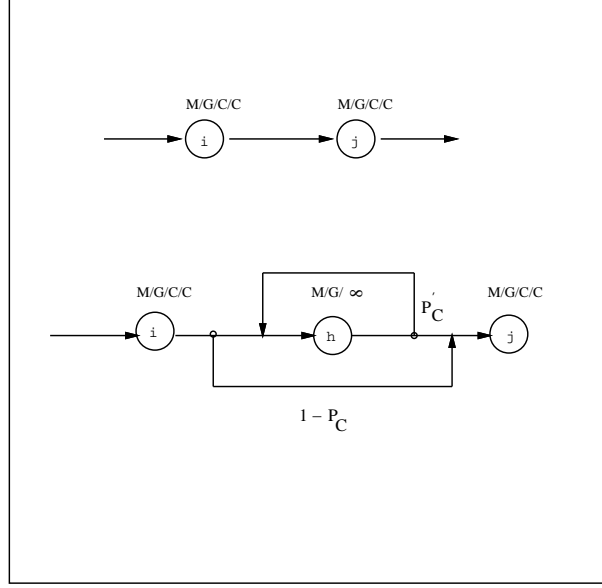
Figure 4: Expansion Method

Stage 2: Parameter Estimation

The value of P_C can be determined from known analytical results, and the value of P'_C is determined from approximation results. For the $M/G/C/C$ state dependent queue, from Equation (1), the probability of being at capacity, C, is given by $P_C = P(n = C)$.

After a customer completes service at a holding node, it is forced to return to this holding node with probability P'_C for another immediate service delay. An approximation, developed previously (Labetoulle and Pujolle, 1980), that uses diffusion techniques to calculate P'_C states that:

$$P'_C = \left\{ \frac{\mu_j + \mu_h}{\mu_h} - \frac{\lambda \left[(r_2^C - r_1^C) - (r_2^{C-1} - r_1^{C-1}) \right]}{\mu_h \left[(r_2^{C+1} - r_1^{C+1}) - (r_2^C - r_1^C) \right]} \right\}^{-1}, \tag{8}$$

where r_1 and r_2 are the roots to:

$$\lambda - (\lambda + \mu_h + \mu_j)x + \mu_h x^2 = 0, \tag{9}$$

where:

$$\lambda = \lambda_j - \lambda_h(1 - P'_C) \tag{10}$$

and

$$\lambda_j = \tilde{\lambda}_i(1 - P_C) = \tilde{\lambda}_i - \lambda_h. \tag{11}$$

Using renewal theory, it can be shown that the service rate distribution of the holding node is as follows:

$$\mu_h = \frac{2\mu_j}{1 + \sigma^2 \mu_j^2}, \tag{12}$$

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where σ^2 is the service time variance.

In $M/G/C/C$ systems, we remind that the service rate μ_j is state dependent but a reasonable assumption is to consider the worst case, which means that $\mu_h = \mu_j \approx C/[E(S)/f(C)]$.

Finally, in those light traffic cases, the blocking probability may became null or very small and λ_j may be underestimated. In order to compensate, an additional term is added to Equation (11), resulting in the following equation:

$$\lambda_j = \tilde{\lambda}_i(1 - P_C) + \lambda_h(1 - P_C')(1 - P_C)^{\rho_i + \rho_j}, \tag{13}$$

where $\rho_i = \lambda_i/\mu_i$ and $\rho_j = \lambda_j/\mu_j$ are the utilizations of the nodes.

Stage 3: Feedback Elimination

A reconfiguration of the holding node is performed, so that the strong dependencies in arrival processes caused by the repeated visits (feedback) to the artificial node are removed. The feedback arc is removed from the holding node and the service rate at this node is recomputed as follows:

$$\mu_h' = (1 - P_C')\mu_h. \tag{14}$$

The average service time that a customer spends at node i preceding node j is given by:

$$\tilde{\mu}_i^{-1} = \mu_i^{-1} + P_C \mu_h'^{-1}. \tag{15}$$

We remark that a feedback elimination is only necessary in those cases in where the performance measure of interest is the sojourn time, making no effect on measures such as the number of customers or throughput.

3.2 Split and Merge

The Expansion Method can be easily extended to splitting and merging topologies. Once demonstrated its use to these basic topologies, any other general network topology could be drawn as a judicious combination.

In splitting topologies, the routing probabilities need to be known. Then, the throughput has only to be decomposed according to the branching probabilities. The capacity C to be used in Equation 8 for all following nodes must be augmented by the number of servers in parallel at the splitting node (Bell, 1982).

Similar changes must to be made in the merging topology cases. In such cases, the arrival rate to the merging node is the sum of the throughput of the previous nodes. The capacity C used in Equation 8 needs to be augmented by the number of parallel servers at each of the previous nodes.

Concerning these changes, it is important to stress that one only could evaluate the performance measurements of a certain node after that moment all nodes whose flows converge to it had the same measures evaluate already.

4 Experiments

We have developed in $C++$ language two classes. The class `Mgcc` implements all basic functions related to a single $M/G/C/C$ queues. The class `MgccNet` extends the previous one, including the Expansion Method for any arbitrary topology. The class headers are presented in the Appendix. The full code is freely available through anonymous ftp at the address `ftp://ftp.est.ufmg.br`.

In order to better show the software capabilities, we shall demonstrate how some of the previous figures were generated, as well as present some numerical results using various topologies.

In Appendix A, we show the codes devoped to compute Figures 2 and 3.

5 Conclusion

We have presented a general methodology suitable for analysis of $M/G/C/C$ systems with state dependent service rates. The importance of this model was stressed and a short review of recent results on the area was presented. We have also discussed in details the application of the model in pedestrian flow related problems. Finally, we have made available, freely, an object-oriented software that could be an important building block in more complex analysis, such as those including optimization problems that embed congestion effects (MacGregor Smith, 1996).

A Computer Codes

```
class Mgcc {
  friend class MgccNet;

// declaring variables for curve adjust
  double k, v1, A, vA, B, vB;

// declaring input variables
  int service;
  double l, w, externLambda;

// declaring auxiliary variables
  int capacity, bufferSize;
  double lambda, gamma, beta, mu_h;

// declaring output variables (performance measures)
  double rho, pBalk, expectedNumber, throughput;
  double lambdaEffective, wait;

public:
   Mgcc(){};
   ~Mgcc(){};

   void AdjustCurve(float maxDensity,
                    float speedLoneOccupant,
                    float densityA, float speedA,
                    float densityB, float speedB);
   void Setup(int servType, float length,
              float width, float lambdaInput);
   void ShowSettings();
   void SetCapacity();
   void SetShapeForm();
   double LinearServiceRate(int customers);
   double ExpServiceRate(int customers);
   void SetPerformance();
   void ShowPerformance();
   void ShowAll();
};
```

Figure 5: Class `Mgcc`

```
class MgccNet: public Mgcc {

   typedef struct NetNodeType {
     int index;
     double probability;
     NetNodeType *next;
   } NetNodeType;

   Mgcc *node;
   NetNodeType **prevNode;
   NetNodeType *endNodes;
   int *nodeStatus;
   int nodeNumber;

public:
   MgccNet(){};
   ~MgccNet();

   void Setup();
   void SetPerformance(int index);
   void EvaluateEnds();
   void ShowData();
};
```

Figure 6: Class `MgccNet`

```
#include <stdlib.h>
#include <stdio.h>
#include "mgcc.h"

void main() {
  float lambda;
  Mgcc mgccExponential;
  mgccExponential.AdjustCurve(5,1.5,2,0.64,4,0.25);
  for (lambda=1; lambda <= 6; lambda += 0.001 ) {
    mgccExponential.Setup(SERV_EXP,8.0,2.5,lambda);
    mgccExponential.SetPerformance();
    mgccExponential.ShowAll();
  }
}
```

Figure 7: Program to Generate Figure 2

```
#include <stdlib.h>
#include <stdio.h>
#include "mgcc.h"

void main() {
  int i;
  Mgcc mgccLinear, mgccExponential;
  mgccLinear.AdjustCurve(5,1.5,0.0,0.0,0.0,0.0);
  mgccLinear.Setup(SERV_LIN,8,2.5,0);
  mgccExponential.Setup(SERV_EXP,8.0,2.5,0);
  mgccExponential.AdjustCurve(5,1.5,2.0,0.64,4.0,0.25);
  mgccExponential.SetShapeForm();
  for (i=0; i < 100; i++ ) {
    fprintf(OUT_FILE, "%d %f %f\n", i+1,
            mgccLinear.LinearServiceRate(i+1),
            mgccExponential.ExpServiceRate(i+1));
  }
}
```

Figure 8: Program to Generate Figure 3

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