A Power Series Expansion for the SN Distribution Function

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Abstract

Univariate skew symmetric models have been considered by several authors and a classical example is the skew normal distribution. The distribution theory literature related to the skew normal distribution has grown rapidly in recent years, and a number of extensions and alternative formulations has been put forward. For the first time, we propose a simple power series expansion for the cumulative distribution function of the skew normal distribution. We also obtain a power series expansion for the quantile function of this distribution. We perform some numerical studies of these series to determine regions where they converge rapidly.

Keywords: Skew normal distribution. Skew normal quantile function. Normal distribution. Normal quantile function. Power series expansion. Owen's function.

1 Introduction

The normal distribution plays a vital role in the statistical analysis of continuous distributions. In fact, in most of the theoretical and applied works, it is assumed that the data follow a normal or nearly normal distribution. For this reason, the extension of the normal distribution to skewed family of distributions has been received considerable attention in recent years. The skew normal (SN) distribution was introduced by O'Hagan and Leonard (1976) and discussed by Azzalini (1985). In this article, we provide a simple elegant power series expansion for the cumulative distribution of the skew normal distribution, which includes the classical expansion for the normal distribution as a particular case. The SN distribution function is usually expressed in terms of Owen's function (Owen 1956). Borth (1973), Young and Minder (1974), Hill (1978) and Thomas (1979) provided different ways to calculate

Owen's function and Patefield and Tandy (2000) presented a detailed study of this function.

A random variable Z has a SN distribution with parameter α, $-\infty < \alpha < \infty$, say $Z \sim SN(\alpha)$, if its probability density function (pdf) has the form

$$f(z;\alpha) = 2\phi(z)\Phi(\alpha z), \quad -\infty < z < \infty, \tag{1}$$

where $\phi(.)$ and $\Phi(.)$ are the standard normal density and distribution functions, respectively. One can verify that the normal distribution is recovered when $\alpha = 0$, and that the absolute value of the skewness increases as the absolute value of α increases. The distribution is right skewed if $\alpha > 0$ and is left skewed if $\alpha < 0$ and it has a number of properties resembling those of the normal distribution.

The cumulative distribution function (cdf) corresponding to (1) is defined by

$$F(z;\alpha) = 2\int_{-\infty}^{z}\int_{-\infty}^{s\alpha} \phi(s)\phi(t)dtds. \tag{2}$$

The calculation of (2) can be obtained from the function $T(z;\alpha)$ studied by Owen (1956), which represents the integral of the standard normal bivariate over a region bounded by the curves $x = z, y = 0$ and $y = x\alpha$ in the plane (x, y). Then,

$$F(z;\alpha) = \Phi(z) - 2T(z;\alpha), \tag{3}$$

where $T(z;\alpha)$ is the Owen's function (also called T-function) defined by

$$T(z;\alpha) = (2\pi)^{-1}\int_0^{\alpha} \frac{\exp\left\{-\frac{1}{2}z^2(1+x^2)\right\}}{1+x^2}dx, \quad (-\infty < z, \alpha < \infty).$$

Owen (1956) presented the following series for the T function

$$T(z;\alpha) = (2\pi)^{-1}\left\{\arctan\alpha - \sum_{j=0}^{\infty} c_j\alpha^{2j+1}\right\},$$

where

$$c_j = (-1)^j\frac{1}{2j+1}\left\{1 - \exp\left(-\frac{1}{2}z^2\right)\sum_{i=0}^{j}\frac{z^{2i}}{2^i i!}\right\}.$$

This series converges rapidly for small z and α, but converges slowly for large z and α close to one.

Patefield and Tandy (2000) discussed different forms to calculate the Owen's function. Indeed, it was presented six different methods. In each method, they used series expansions which rely on the region (z, α) of the plane to obtain accurate values for $T(z; \alpha)$. However, the computation of the cdf (3) is complicated according to the region (z, α). In this article, we obtain a very simple way to calculate the cdf of the SN distribution.

The rest of the paper is organized as follows. In Section 2, we provide some basic relations between the SN cdf and Owen's function. In Section 3, we give a simple power series expansion for the cdf of the SN distribution. Section 4 is devoted to the quantile function and Section 5 gives some numerical computation of the series. Conclusion remarks are presented in Section 6.

2 Some Basic Relations

Lemma 1. *The required range of evaluation of the function $T(z; \alpha)$ can be reduced from $-\infty < z, \alpha < \infty$ to $0 < \alpha \leq 1$ and $z \geq 0$ by using in turn the following relations due to Owen (1956)*

$$\begin{aligned} T(z; -\alpha) &= -T(z; \alpha), \quad T(-z; \alpha) = T(z; \alpha) \quad \textit{and} \\ T(z; \alpha) &= \tfrac{1}{2}\{\Phi(z) + \Phi(\alpha z)\} - \Phi(z)\Phi(\alpha z) - T(\alpha z; 1/\alpha). \end{aligned} \tag{4}$$

Lemma 2. *The SN cdf for $\alpha \geq 1$ is given by*

$$F(z; \alpha) = 2\Phi(z)\Phi(\alpha z) - F\left(\alpha z; \frac{1}{\alpha}\right).$$

Proof. If we assume that $T(z, \alpha)$ holds for $0 < \alpha \leq 1$, we can express $T(z, \alpha)$ for $\alpha \geq 1$ as in the last equation of (4). By inserting it in (3) yields

$$F(z; \alpha) = \Phi(z) - 2T(\alpha z; \alpha) = -\Phi(\alpha z) + 2\Phi(z)\Phi(\alpha z) + 2T(\alpha z; 1/\alpha).$$

Using again (3) for $F(\alpha z; 1/\alpha)$, we have $2T(\alpha z; 1/\alpha) = \Phi(\alpha z) - F(\alpha z; 1/\alpha)$ and then by inserting in the last equation, the lemma follows. In the special case $\alpha = 1$, we obtain $F(z; 1) = \Phi(z)^2$.

3 A Simple Expansion for the Distribution Function

Theorem 1. *Let Z be a $SN(\alpha)$ random variable. For $0 < \alpha < 1$, the cdf of Z can be expressed as*

$$F(z;\alpha) = \frac{1}{2} - \frac{1}{\pi}\arctan(\alpha) + \sum_{r=1}^{\infty} c_r z^r, \tag{5}$$

where the coefficients $c_r = c_r(\alpha)$ as functions of α are defined at the end of the proof.

Proof.
Henze (1986) demonstrated that the cdf of Z can be written as

$$P(Z \leq z) = 2\int_0^{\infty} \Phi\left(\frac{z - pu}{q}\right)\phi(u)du, \tag{6}$$

where

$$p = p(\alpha) = \frac{\alpha}{\sqrt{1+\alpha^2}} \quad \text{and} \quad q = q(\alpha) = \frac{1}{\sqrt{1+\alpha^2}}.$$

Simple differentiation of (6) yields (1).

We use (6) to demonstrate (5). We begin with the well-known power series expansion for the standard normal cdf

$$\Phi(y) = \frac{1}{2}\left\{1 + \mathrm{erf}\left(\frac{\mathrm{y}}{\sqrt{2}}\right)\right\},$$

where the error function erf(.) is

$$\mathrm{erf}(\mathrm{x}) = \frac{2}{\sqrt{\pi}}\sum_{\mathrm{k}=0}^{\infty}\frac{(-1)^{\mathrm{k}}\mathrm{x}^{2\mathrm{k}+1}}{(2\mathrm{k}+1)\mathrm{k}!}, \quad |\mathrm{x}| < \infty.$$

Hence, we obtain

$$\Phi\left(\frac{z - pu}{q}\right) = \frac{1}{2}\left\{1 + \mathrm{erf}\left(\frac{\mathrm{z} - \mathrm{pu}}{\mathrm{q}\sqrt{2}}\right)\right\}$$

and then

$$\Phi\left(\frac{z - pu}{q}\right) = \frac{1}{2} + \sum_{k=0}^{\infty} a_k (z - pu)^{2k+1},$$

where the coefficients a_k are given by

$$a_k = a_k(\alpha) = \frac{(-1)^k}{\sqrt{\pi}(2k+1)k!\,(q\sqrt{2})^{2k+1}}.$$

The binomial expansion leads to

$$\Phi\left(\frac{z-pu}{q}\right) = \frac{1}{2} + \sum_{k=0}^{\infty} a_k \sum_{j=0}^{2k+1} \binom{2k+1}{j} (-1)^{j+1} p^{2k+1-j} z^j u^{2k+1-j}. \tag{7}$$

Inserting (7) into (6) yields

$$\begin{aligned} F(z;\alpha) \;&=\; \int_0^{\infty} \phi(u)du + \sum_{k=0}^{\infty} a_k \sum_{j=0}^{2k+1} \binom{2k+1}{j} (-1)^{j+1} p^{2k+1-j} \\ & \quad \times z^j \int_0^{\infty} 2u^{2k+1-j} \phi(u)du. \end{aligned}$$

Defining

$$G(m) = 2\int_0^{\infty} x^m \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)dx = \frac{2^{m/2}}{\sqrt{\pi}}\,\Gamma\left(\frac{m+1}{2}\right),$$

where $\Gamma(\cdot)$ is the gamma function, we can write

$$F(z;\alpha) = \frac{1}{2} + \sum_{k=0}^{\infty} \sum_{j=0}^{2k+1} B(k,j) z^j. \tag{8}$$

Here,

$$B(k,j) = (-1)^{j+1} \binom{2k+1}{j} p^{2k+1-j}\, a_k\, G(2k+1-j), \tag{9}$$

and a_k and $G(2k+1-j)$ are defined before.

Changing the sums $\sum_{k=0}^{\infty} \sum_{j=0}^{2k+1} B(k,j) z^j$ by

$$\sum_{k=0}^{\infty} B(k,0) + \delta_j \sum_{j=1}^{\infty} \sum_{k=(j/2)-1}^{\infty} B(k,j)z^j + (1-\delta_j) \sum_{j=1}^{\infty} \sum_{k=(j-1)/2}^{\infty} B(k,j)z^j,$$

where $\delta_j = 1$ if j is even and $\delta_j = 0$ if j is odd, we can write

$$F(z;\alpha) = \frac{1}{2} + \sum_{k=0}^{\infty} B(k,0) + \sum_{r=0}^{\infty}\sum_{k=r}^{\infty} B(k,2r+1)z^{2r+1} + \sum_{r=0}^{\infty}\sum_{k=r}^{\infty} B(k,2r)z^{2r}. \tag{10}$$

Hence, a power series expansion for the SN cdf can be expressed as

$$F(z;\alpha) = \frac{1}{2} + c_0 + \sum_{r=1}^{\infty} c_r z^r,$$

where the coefficients $c_0 = c_0(\alpha)$ and $c_r = c_r(\alpha)$ are given by

$$c_0 = \sum_{k=0}^{\infty} B(k,0), \qquad c_{2r+1} = \sum_{k=r}^{\infty} B(k,2r+1), \quad c_{2r} = \sum_{k=r}^{\infty} B(k,2r),\ \forall r \geq 0. \tag{11}$$

For the coefficient c_0, we obtain from (11)

$$\begin{aligned} c_0 &= \sum_{k=0}^{\infty} B(k,0) = \sum_{k=0}^{\infty} \binom{2k+1}{0}(-1)a_k\, p^{2k+1}G(2k+1) \\ &= \sum_{k=0}^{\infty}(-1)^{k+1}\frac{p^{2k+1}(\sqrt{2})^{2k+1}\Gamma(k+1)}{\pi(2k+1)k!(\sqrt{2})^{2k+1}q^{2k+1}} = \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^{k+1}\frac{k!}{(2k+1)k!}\left(\frac{p}{q}\right)^{2k+1}. \end{aligned}$$

Since $\alpha = p/q$, we have

$$c_0 = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}\alpha^{2k+1} = -\frac{1}{\pi}\arctan(\alpha) \tag{12}$$

The coefficients c_{2r+1} and c_{2r} in (11) reduce to

$$c_{2r+1} = \frac{(1+\alpha^2)^{r+1/2}}{\alpha^{2r}\pi\, 2^{r+1/2}}\sum_{k=r}^{\infty}\frac{(-1)^k\binom{2k+1}{2r+1}\alpha^{2k}\Gamma\left((k-r)+1/2\right)}{(2k+1)k!}. \tag{13}$$

and

$$c_{2r} = \frac{(1+\alpha^2)^{r}}{\alpha^{2r-1}\pi\, 2^{r}}\sum_{k=r}^{\infty}\frac{(-1)^{k+1}\binom{2k+1}{2r}\alpha^{2k}\Gamma\left((k-r)+1\right)}{(2k+1)k!}. \tag{14}$$

Equations (5), (12), (13) and (14) are the main results of this section. The theorem is then proved.

Remark 1. *As a special case, we obtain the normal cumulative function if* $\alpha \to 0$. *We have* $\lim_{\alpha\to 0} c_0(\alpha) = 0$. *For calculating* c_{2r} *and* c_{2r+1}, *we obtain*

$$B(k, 2r) = -\binom{2k+1}{2r} p^{2(k-r)+1} a_k G(2(k-r)+1)$$

and

$$B(k, 2r+1) = \binom{2k+1}{2r+1} p^{2(k-r)} a_k G(2(k-r)),$$

where a_k *and* $G(2(k-r))$ *are given before. For* $k \geq r$, $B(k, 2r)$ *vanishes and, for* $k > r$, $B(k, 2r+1)$ *vanishes since* $\lim_{\alpha\to 0} p(\alpha) = 0$. *For the case* $k = r$, $\lim_{\alpha\to 0} p(\alpha) = 0$, $\lim_{\alpha\to 0} q(\alpha) = 1$ *and* $G(0) = 1$. *So, we have*

$$\lim_{\alpha\to 0} c_{2r+1}(\alpha) = a_r G(0) \lim_{\alpha\to 0} p(\alpha)^{k-r} = (-1)^r \{\sqrt{\pi}(2r+1) r!\, (\sqrt{2})^{2r+1}\}^{-1},$$

where we used the result $\lim_{\alpha\to 0} p(\alpha)^0 = 1$. *Taking the limit when* $\alpha \to 0$, *Eq. (5) yields the classical expansion for the normal cdf*

$$\lim_{\alpha\to 0} F(z;\alpha) = F(z;0) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+1}}{(2r+1) r!\, 2^{(2r+1)/2}}. \tag{15}$$

Corollary 1. *Let* Z *be a* $SN(\alpha)$ *random variable.*

1. *For* $\alpha \geq 1$, *the cdf of* Z *can be expressed as*

$$F(z;\alpha) = 2\Phi(z)\Phi(\alpha z) - F(\alpha z; \alpha^{-1}),$$

   where $F(\alpha z; \alpha^{-1})$ *is given by Theorem 1.*

2. *For* $\alpha = 1$, *the cdf of* Z *can be written as*

$$F(z;1) = \sum_{n=0}^{\infty} h_n z^n,$$

   where the coefficients h_n *(*$n = 0, 1, \cdots$*) are defined by*

$$h_0 = \frac{1}{4} + \frac{1}{2\pi}, \quad h_{2n+1} = \frac{2(-1)^n}{\sqrt{\pi}(2n+1) n!\, (\sqrt{2})^{2n+1}}$$

   and

$$h_{2n} = \frac{(-1)^n}{\pi\ 2^n} \sum_{m=0}^{2n} \frac{1}{(2m+1) m! [2(n-m)+1](n-m)!}.$$

Proof. We only need to use the above results in each case.

4 Quantile Function

The quantile function (qf) is defined by $q = F^{-1}(p;\alpha) = Q(p;\alpha)$, where $F(p;\alpha)$ follows (5). We shall use the Lagrange theorem (Markushevich, 1965, vol 2, pp. 88) to obtain the expansion for the quantile function. If we assume that the power series expansion holds

$$w = F(z) = w_0 + \sum_{n=1}^{\infty} f_n (z - z_0)^n, \qquad f_1 = F'(z) \neq 0,$$

where $F(z)$ is analytic at a point z_0 that gives a simple w_0-point. Then, the inverse function $F^{-1}(w)$ exists and is single-valued in the neighborhood of the point $w = w_0$. The power series inverse $z = Q(w)$ is given by

$$z = Q(w) = z_0 + \sum_{n=1}^{\infty} g_n (w - w_0)^n,$$

where

$$g_n = \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \left\{ [\psi(z)]^n \right\} \Big|_{z=z_0} \quad \text{and} \quad \psi(z) = \frac{z - z_0}{F(z) - w_0}.$$

Theorem 2. *Let Z be a $SN(\alpha)$ random variable. For $0 < \alpha < 1$, the qf of Z can be expressed as*

$$Q(w;\alpha) = \sum_{n=1}^{\infty} g_n \left[w - \arctan(\alpha^{-1}) \right]^n,$$

where the coefficients $g_n = g_n(\alpha)$ are functions of α defined at the end of the proof.

Proof.
We now rearrange expansion (5)

$$F(z;\alpha) = \frac{1}{2} + c_0 + z[c_1 + c_2 z + c_3 z^2 + \ldots].$$

Setting $f_n = f_n(\alpha) = c_{n+1}$ for $n = 0, 1, 2, \ldots$, the power series expansion for $F(z)$ becomes

$$F(z;\alpha) = \frac{1}{2} + c_0 + z \sum_{n=0}^{\infty} f_n z^n.$$

Setting $z_0 = 0$ and $w_0 = (1/2 + c_0) = \frac{1}{\pi}\arctan(\alpha^{-1})$, we define

$$\psi(z) = \frac{z}{F(z;\alpha) - (\frac{1}{2} + c_0)} = \frac{1}{\sum_{n=0}^{\infty} f_n z^n}.$$

We can obtain the inverse of the power series $\sum_{n=0}^{\infty} f_n z^n$ using equation (0.313) from Gradshteyn and Ryzhik (2000). We have

$$\psi(z) = \frac{1}{\sum_{n=0}^{\infty} f_n z^n} = \frac{1}{f_0}\sum_{n=0}^{\infty} d_n z^n,$$

so that d_n can be calculated recursively from the coefficients in (13) and (14) by

$$d_0 = 1 \quad \text{and} \quad d_n = -\frac{1}{c_1}\sum_{k=1}^{n} c_{k+1} d_{n-k}, \; n \geq 1$$

and

$$f_0 = c_1 = \frac{(1+\alpha^2)^{1/2}}{\sqrt{2}\pi}\sum_{k=0}^{\infty}\frac{(-1)^k \alpha^{2k}\Gamma\left(k+1/2\right)}{(2k+1)k!}. \tag{16}$$

We can obtain $\psi(z)^n$ from (4)

$$\psi(z)^n = \left[\frac{1}{c_1}\sum_{i=0}^{\infty} d_i z^i\right]^n.$$

We now use an equation of Gradshteyn and Ryzhik (2000) (Section 0.314) for power series raised to powers. For any n positive integer, we can write

$$\psi(z)^n = \left[\frac{1}{c_1}\sum_{i=0}^{\infty} d_i z^i\right]^n = \frac{1}{c_1^n}\sum_{i=0}^{\infty} c_{i,n} z^i,$$

where the coefficients $c_{i,n}$ (for $i = 1, 2, \cdots$) can be easily obtained from the recurrence relation

$$c_{i,n} = \frac{1}{i}\sum_{m=1}^{i}(nm - i + m) d_m c_{i-m,n},$$

and $c_{0,n} = d_0^n = 1$. The coefficient $c_{i,n}$ can be obtained from $c_{0,n}, \ldots, c_{i-1,n}$ and therefore from $d_0, \ldots, d_i$. Clearly, $c_{i,n}$ can be given explicitly in terms of the quantities d_i, although it is not necessary for programming numerically our expansions in any algebraic or numerical software. The power series with the first $(n+1)$ terms can be written as

$$\psi(z)^n = \frac{1}{c_1^n}(c_{0,n} + c_{1,n}z + \cdots + c_{n-1,n}z^{n-1} + c_{n,n}z^n + \cdots)$$

The derivative of order $(n-1)$ is given by

$$\frac{d^{n-1}}{dz^{n-1}}\{[\psi(z)]^n\}\bigg|_{z=0} = \frac{(n-1)!c_{n-1,n}}{c_1^n},$$

and then

$$g_n = g_n(\alpha) = \frac{1}{n!}\frac{d^{n-1}}{dz^{n-1}}\{[\psi(z)]^n\}\bigg|_{z=0} = \frac{c_{n-1,n}}{nc_1^n}.$$

Hence, a power series for the qf can be written as

$$Q(w) = \sum_{n=1}^{\infty} g_n \left[w - \frac{1}{\pi}\arctan(\alpha^{-1})\right]^n.$$

In a similar manner, we can write the qf of the SN distribution as

$$Q(w) = \sum_{k=0}^{\infty} G(k,\alpha)w^k,$$

where

$$G(k,\alpha) = \sum_{n=k+1}^{\infty} (-1)^{n-k}\binom{n}{k} g_n(\alpha)\left[\frac{1}{\pi}\arctan(\alpha^{-1})\right]^{n-k}.$$

Corollary 2. *Let Z be a standard normal random variable. The qf of Z is given by*

$$Q(w) = \sum_{n=1}^{\infty} g_n \left(w - \frac{1}{2}\right)^n,$$

where the coefficient g_n are constants defined by the Lagrange theorem.

Proof.
The cdf expansion (15) can be rewritten as

$$F(z) = \frac{1}{2} + z(c_1 + c_3z^2 + c_5z^4 + \cdots) \quad \text{with} \quad c_{2r+1} = \frac{(-1)^r}{\sqrt{\pi}\,(2r+1)r!2^{(2r+1)/2}}.$$

Let $f_n = c_{n+1}\delta_n$ for $n = 0, 1, 2, \ldots$, where δ_n is the Kronecker delta: $\delta_n = 1$ if n is even and $\delta_n = 0$ is n is odd. For $n = 0$, we have $f_0 = c_1 = \frac{1}{\sqrt{2\pi}}$. Hence, the power series expansion for $F(z)$ becomes

$$F(z) = \frac{1}{2} + z \sum_{n=0}^{\infty} f_n z^n. \tag{17}$$

Setting $z_0 = 0$ and $w_0 = \frac{1}{2}$, we define

$$\psi(z) = \frac{z}{F(z) - \frac{1}{2}} = \frac{1}{\sum_{n=0}^{\infty} f_n z^n}.$$

We can obtain the inverse of the power series $\sum_{n=0}^{\infty} f_n z^n$ using equation (0.313) from Gradshteyn and Ryzhik (2000). We have

$$\psi(z) = \frac{1}{\sum_{n=0}^{\infty} f_n z^n} = \frac{1}{c_1} \sum_{n=0}^{\infty} d_n z^n = \sqrt{2\pi} \sum_{n=0}^{\infty} d_n z^n, \tag{18}$$

so that d_n can be calculated recursively from the coefficients in (5) by

$$d_0 = 1 \quad \text{and} \quad d_n = -\frac{1}{c_1} \sum_{k=1}^{n} d_{n-k} c_{k+1} \delta_k, \; n \geq 1.$$

We can obtain $\psi(z)^n$ from (18)

$$\psi(z)^n = (\sqrt{2\pi})^n \left(\sum_{i=0}^{\infty} d_i z^i \right)^n.$$

We now use an equation of Gradshteyn and Ryzhik (2000) (Section 0.314) for power series raised to powers. For any n positive integer, we can write

$$\psi(z)^n = (\sqrt{2\pi})^n \left(\sum_{i=0}^{\infty} d_i z^i \right)^n = (\sqrt{2\pi})^n \sum_{i=0}^{\infty} c_{i,n} z^i,$$

where the coefficients $c_{i,n}$ for $i = 1, 2, \ldots$ can be easily obtained from the recurrence relation

$$c_{i,n} = \frac{1}{i} \sum_{m=1}^{i} (nm - i + m) d_m c_{i-m,n},$$

and $c_{0,n} = d_0^n = 1$. The coefficient $c_{i,n}$ follows from $c_{0,n}, \ldots, c_{i-1,n}$ and therefore from $d_0, \ldots, d_i$. The power series with the first $(n+1)$ terms can be written as

$$\psi(z)^n = (\sqrt{2\pi})^n (c_{0,n} + c_{1,n} z + \cdots + c_{n-1,n} z^{n-1} + c_{n,n} z^n + \cdots).$$

The derivative of order $(n-1)$ is given by

$$\frac{d^{n-1}}{dz^{n-1}} \left\{ [\psi(z)]^n \right\} \bigg|_{z=0} = (n-1)! (\sqrt{2\pi})^n c_{n-1,n},$$

and then

$$g_n = \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \left\{ [\psi(z)]^n \right\} \bigg|_{z=0} = \frac{(\sqrt{2\pi})^n c_{n-1,n}}{n}.$$

Hence, the power series quantile function reduces to

$$Q(w) = \sum_{n=1}^{\infty} g_n \left(w - \frac{1}{2} \right)^n.$$

An alternative expression for the qf of the SN distribution is

$$Q(w) = \sum_{k=0}^{\infty} G(k, \alpha) w^k,$$

where

$$G(k, \alpha) = \sum_{n=k+1}^{\infty} (-1)^{n-k} \binom{n}{k} g_n \left(\frac{1}{2} \right)^{n-k}.$$

5 Numerical Computation of the Series

5.1 Computation of $F(z, \alpha)$

Eq. (8) can be easily evaluated with a software such as Matlab or R (R Development Core Team, 2009). The serie can be implemented without direct calculation of the Γ functions and binomial coefficients. Using $\Gamma(z+1) = z\Gamma(z)$ and $\binom{n}{k+1} = \binom{n}{k} \frac{n-k}{k+1}$, we have $G(m+1) = mG(m-1)$ and the following recurrence relations for $B(k, j)$:

$$\frac{B(k+1, j)}{B(k, j)} = -\alpha^2 \left(\frac{2k+1}{2k+3-j} \right)$$

and

$$\frac{B(k,j+2)}{B(k,j)} = \left(\frac{1+\alpha^2}{\alpha^2}\right)\frac{2k+1-j}{(j+1)(j+2)},$$

where, using (9) we have, $B(0,0) = -\alpha/\pi$ and $B(0,1) = [(1+\alpha^2)/2\pi]^{1/2}$. In order to verify the convergence of the series (8) and (10), we implemented the truncated sums

$$F_1(z;\alpha,n) = \frac{1}{2} + \sum_{k=0}^{n}\sum_{j=0}^{2k+1} B(k,j)z^j.$$

and

$$F_2(z;\alpha,n,m) = \frac{1}{2}+\sum_{k=0}^{n} B(k,0)+\sum_{r=0}^{n}\sum_{k=r}^{m} B(k,2r+1)z^{2r+1}+\sum_{r=1}^{n}\sum_{k=r}^{m} B(k,2r)z^{2r}$$

It is easy to verify that for $m = n$, the expressions for $F_1(z;\alpha,n)$ and $F_2(z;\alpha,n,m)$ are identical. The series converges fast for $|z| \leq 3$ and $|\alpha| \leq 0.5$. In this region, we obtain for $F_1(z;\alpha,n)$ an relative error <1e–10 for $n = 120$. Table 1 gives values for $F_1(z;\alpha,n = 120)$ and $F(z;\alpha)$. The missing values in this table correspond to cases where the convergence is not achieved. The best numerical results are obtained using $F_2(z;\alpha,n,m)$ instead of $F_1(z;\alpha,n)$ (see Table 2). For $|z| \leq 3$ and $|\alpha| \leq 0.5$, we obtain an relative error <1e–11 using $F_2(z;\alpha,n,m)$ with $n = 120$ and $m = 20$.

5.2 The Quantile Function

In order to compare the numerical results of the series with the exact values, we use the R-base package or a very simple method to compute the values of the quantiles, similar to a root-finding algorithm secant method:

Numerical method to determine $Q(w)$ given the cdf $F(z)$:
Input: w, number N of iterations, initial interval $[a,b]$, $a < b$
1. set $i = 1$, $x_1 = a$, $x_2 = b$
2. *while* $(i \leq N)$
3. $\quad w_i$=*mean* (x_1, x_2)
4. $\quad$ if$(F(w_i) > w)$ *do* $x_2 = w_i$
5. $\quad$ else $x_1 = w_i$
6. $\quad i \leftarrow i+1$
7. end *while*

Table 1: Results of series $F(z;\alpha)$ and $F_2(z;\alpha,n)$ for $z=\pm 3$ and different values of α.

	α	$F(z;\alpha)^*$	$F_2(z;\alpha,n=120)$	Relative Error
	0.50	0.997439859211200	0.997439859211200	<1.00e–15
	0.60	0.997369579953320	0.997369579953310	9.68e–15
	0.70	0.997331886928160	0.997331886928480	-3.21e–13
$z=3$	0.80	0.997313505649600	0.997313502668940	2.99e–09
	0.85	0.997308555317360	0.997318822197400	-1.03e–05
	0.90	0.997305338324570	1.191330695776200	-1.95e–01
	0.95	0.997303294961250	-	-
	0.50	0.000139655274460	0.000139655274472	-8.94e–11
	0.60	0.000069376016579	0.000069376016751	-2.48e–09
	0.70	0.000031682991424	0.000031675144105	2.48e–04
$z=-3$	0.80	0.000013301712867	-	-
	0.85	0.000008351380616	-	-
	0.90	0.000005134387833	-	-
	0.95	0.000003091024513	-	-

* Using $F(z;\alpha)=\Phi(z)-2T(z,\alpha)$

For the quantile expansion $Q(w)$ of the normal cdf, we make a comparison between (2) and the Steinbrecher and Shaw (2007) expansion

$$Q(u)=\sum_{p=0}^{\infty}\frac{w_p}{(2p+1)}[\sqrt{2\pi}(u-1/2)]^{2p+1}, \tag{19}$$

where $w_{p+1}=\frac{1}{2}\sum_{j=0}^{p}\frac{w_j w_{p-j}}{(j+1)(2j+1)}$ and $w_0=1$. Figure 1 shows the behavior of the series (2) and (19) in the tail region of the normal cdf ($w>0.985$).

5.3 Computation of the Quantile Function of the Skew Normal

For the SN quantile function, we evaluate the convergence of

$$Q(w;\alpha,m)=\sum_{n=1}^{m}g_n(\alpha)\,(w-w_0)^n\,, \tag{20}$$

where $w_0=\frac{1}{\pi}\arctan(\alpha^{-1})$. This serie converges fast for small values of α. Using $m=150$, we obtain the largest absolute error <1e–10 in $0.1\le\alpha\le 0.4$

Table 2: Results of series $F(z;\alpha)$ and $F_1(z;\alpha,n,m)$ for $z = \pm 3$ and different values of α.

	α	$F(z;\alpha)^*$	$F_1(z;\alpha, n=120, m=20)$	Relative Error
	0.50	0.997439859211200	0.997439859211200	<1.00e–15
	0.60	0.997369579953320	0.997369579953400	-8.03e–14
	0.70	0.997331886928160	0.997331886882800	4.55e–11
$z=3$	0.80	0.997313505649600	0.997313490923360	1.48e–08
	0.85	0.997308555317360	0.997308438157650	1.17e–07
	0.90	0.997305338324570	0.997305139857720	1.99e–07
	0.95	0.997303294961250	0.997313379740450	-1.01e–05
	0.50	0.000139655274460	0.000139655274461	-6.59e–12
	0.60	0.000069376016579	0.000069376016681	-1.47e–09
	0.70	0.000031682991424	0.000031683055923	-2.04e–06
$z=-3$	0.80	0.000013301712867	0.000013304994166	-2.47e–04
	0.85	0.000008351380616	0.000008309054223	5.07e–03
	0.90	0.000005134387833	0.000004078414077	2.06e–01
	0.95	0.000003091024513	-0.000014789256413	5.78e+00

* Using $F(z;\alpha) = \Phi(z) - 2T(z,\alpha)$

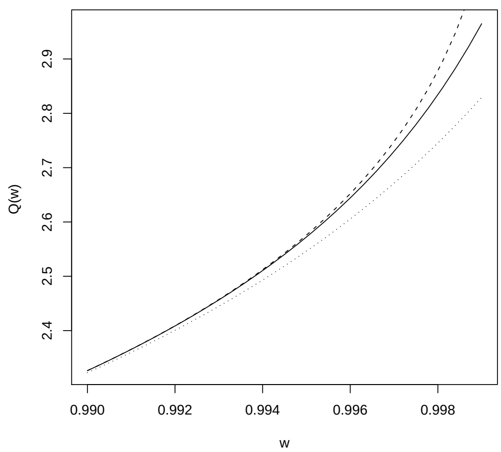

Figure 1: Quantil function $Q(w)$ for: (a) $0.98 \leq w \leq 0.99$ (solid line) (b) using the first 150 terms of the expansion (2) (dotted line) (c) using the first 150 terms of the Steinbrecher-Shaw expansion (19) (dashed line).

and $0.1 \leq z \leq 0.9$. For some values of w in the region $w > 0.6$ and $\alpha > 0.5$, the convergence is very slow. Figure 2 shows $Q(w, \alpha, m)$ for $0.001 \leq w \leq 0.05$ and $\alpha = 0.3$. Tables 3 and 4 give $Q(w; \alpha, m = 150)$ for some values of w and α. The missing values correspond to cases where the convergence is not achieved.

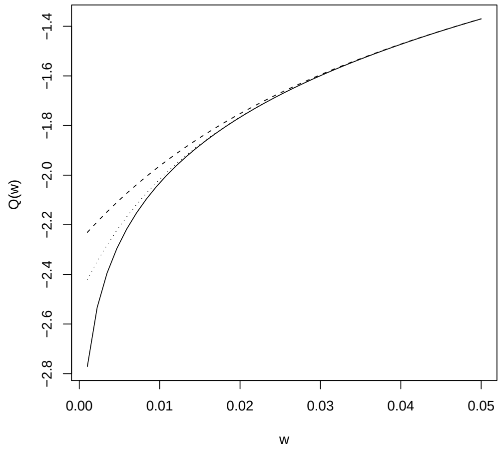

Figure 2: Quantil function of the skew normal distribution for: (a) $\alpha = 0.3$ and $0.001 \leq w \leq 0.05$ (solid line) (b) using the expansion (20) with $m = 40$ (dashed line) and (c) using $m = 80$ (dotted line).

Table 3: Relative errors of the quantil function $Q(w; \alpha, m = 150)$ for some values of α and w.

$\alpha \downarrow$	$w = 0.01$	$w = 0.1$	$w = 0.2$	$w = 0.3$	$w = 0.4$	$w = 0.5$
0.05	3.96e–07	1.92e–07	7.96e–08	2.97e–08	1.00e–08	3.07e–09
0.10	1.68e–12	3.41e–13	4.78e–14	4.36e–15	1.80e–15	-2.12e–15
0.20	-1.33e–16	4.38e–16	1.65e–16	-3.76e–16	8.74e–16	-1.55e–15
0.30	-2.15e–16	2.50e–16	-3.07e–16	-7.88e–16	6.74e–16	-2.05e–15
0.40	-6.79e–16	-3.21e–16	-1.03e–15	-9.75e–15	-5.28e–16	1.43e–15
0.50	3.37e–14	1.00e–16	3.55e–16	8.51e–16	1.32e–15	1.26e–15
0.60	6.37e–16	6.69e–16	6.83e–16	1.17e–15	4.14e–16	5.65e–16
0.70	-6.26e-16	1.11e–15	-1.65e–16	6.01e–16	1.10e–10	4.16e–04
0.80	1.31e-16	9.67e–16	1.14e–11	3.79e–05	-	-

Table 4: Relative errors of the quantil function $Q(w;\alpha,m=150)$ for some values of α and w.

$\alpha\downarrow$	$w=0.6$	$w=0.7$	$w=0.8$	$w=0.9$	$w=0.95$	$w=0.99$
0.05	8,57e–10	2,19e–10	5,16e–11	7,86e–12	-1,38e–07	-
0.1	8,84e–16	1,48e–15	-1,47e–15	-4,98e–12	-2,68e–07	-
0.2	-4,65e–16	1,33e–15	-2,15e–15	-9,11e–12	-4,03e–07	-1,10e–003
0.3	-1,75e–15	9,11e–15	1,46e–14	1,43e–11	7,85e–07	2,92e–003
0.4	1,12e–15	2,47e–16	1,86e–15	-4,21e–12	-1,52e–07	-
0.5	8,23e–16	1,00e–16	2,28e–15	3,16e–09	3,00e–06	-
0.6	1,44e–11	4,48e–05	-	-	-	-
0.7	-	-	-	-	-	-

The convergence of the coefficients $c_{2r}(\alpha)$ and $c_{2r+1}(\alpha)$ in (13) and (14) is very fast. For $w>0.6$ approximately, the convergence of $Q(w;\alpha,m)$ is strongly associated with the behavior of the terms $(w-w_o)^n$ in (20). Figure 3 (a) shows $Q(w;\alpha,m)=\sum_{n=1}^{m} g_n(\alpha)\,(w-w_0)^n$ *versus* m, for $w=0.60$ and $\alpha=0.7$, whereas Figure 3 (b) shows that for $\alpha=0.7$ but with $w=0.65$. In this case, the convergence is not achieved.

(a) (b)

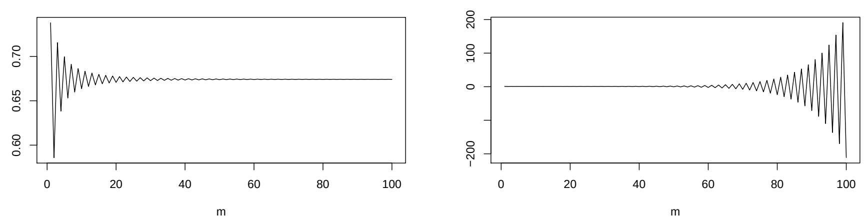

Figure 3: Graphics of convergence and not convergence the quantile function of the skew-normal distribution when: (a) $w=0.60$, $\alpha=0.7$ and (b) $w=0.65$, $\alpha=0.7$, respectively.

6 Conclusions

We provide a new power series expansion for the cumulative distribution of the skew normal (SN) distribution. From this expansion, we obtain as a special case the classical power series expansion for the cumulative distribution of the normal distribution. Following our approach, we derive a power series expansion for the quantile function of the SN distribution. We provide some numerical studies on the adequacy of both expansions.

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