IMA Journal of Mathematics Applied in Business & Industry (1998) **9**, 347–354

Age-reduction models for imperfect maintenance

N. JACK

Mathematical Sciences Division, School of Computing, University of Abertay Dundee, Dundee, Scotland

Maintenance of a deteriorating system is often imperfect, with the state of the system after maintenance being at a level somewhere between new and its prior condition. In this paper, the concept of reduction in virtual or effective age is used to model the effect of both imperfect corrective maintenance (CM) and imperfect preventive maitnenance (PM). Results from counting-process theory then produce a likelihood function necessary for parameter estimation, and the method is tested on known maintenance data. Finally, it is shown how to evaluate, by simulation, the expected number of system failures up to time t under a given periodic PM strategy. This measure is incorporated into a cost rate function which is then minimized to find the optimal length of a PM interval and the optimal number of PMs to carry out before system replacement.

1. Introduction

A system is observed over time $t \geqslant 0$. At each system failure, corrective maintenance (CM) or repair is performed, and preventive maintenance (PM) is also carried out, on occasions to try and reduce system deterioration. Both types of maintenance action take negligible time to complete and are imperfect in the sense that the system's virtual or effective age after maintenance lies somewhere between zero (good-as-new) and its prior level (bad-as-old).

Kijima *et al.* (1988) first used the concept of reduction in virtual age to model the behaviour of a system with imperfect CM, no PM, and periodic replacement. Liu *et al.* (1995) modelled imperfect periodic PM using the concept of virtual age reduction, but assumed only minimal or bad-as-old CM. However, both these papers were optimization-based and did not deal with parameter-estimation problems. Jack (1997) developed statistical techniques for analysing event data from a system subject to minimal CM and known, but not necessarily periodic, imperfect PM.

In this paper, the effects of both types of imperfect maintenance are modelled using the age-reduction concept. Different age-reduction 'factors' $\delta_{\rm CM}$ and $\delta_{\rm PM}$ are used for the two types of maintenance, and two alternative models are proposed.

In model I, it is assumed that:

(i) the system's virtual age after the present CM action equals its virtual age after the previous maintenance action (CM or PM) plus $\delta_{\rm CM}$ times the chronological age accumulated since that previous maintenance;

(ii) the system's virtual age after the present PM action equals its virtual age after the previous PM action plus $\delta_{\rm PM}$ times the virtual age accumulated since the previous PM.

Note that, in the case of age reduction at PM, $\delta_{\rm PM}$ is applied to virtual age accumulated since it is possible that intermediate CM has reduced the corresponding chronological age.

Alternatively, in model II, it is assumed that

(i) the system's virtual age after the present CM action equals δ_{CM} times its virtual age before the present CM;
(ii) the system's virtual age after the present PM action equals δ_{PM} times its virtual age before the present PM.

At CM, only the small part of the system that has failed is repaired or replaced. At PM, however, repairs and replacements can be carried out on several system parts, and it is therefore assumed that, in general, $\delta_{PM} < \delta_{CM}$. Note also the two extreme values for the age-reduction factors: $\delta_{PM} = 0$, which implies perfect (good-as-new) PM, and $\delta_{CM} = 1$ which implies minimal (bad-as-old) CM.

Given information on the occurrence times of previous CM and PM, the system's virtual age at any given time is then determined, and it is shown how to construct a likelihood function to provide estimates (with confidence intervals) of the relevant system lifetime distribution parameters, and the age reduction factors δ_{CM} and δ_{PM}, for both models. An example with known maintenance data is given to illustrate the method, and it is also shown how to use simulation techniques to estimate the system's expected failure count function and then find optimal periodic PM and replacement intervals.

2. Age reduction models

Suppose that the system is observed over k PM intervals. The following notation is needed for both models

t_{ij} = time of ith failure in the jth PM interval ($j = 1, ..., k$; $i = 1, ..., n_j$),
t_{0j} = time of $(j-1)$th PM ($t_{01} = 0$ and $t_{0j} = t_{n_{j-1}+1,j-1}$; $j = 2, ..., k$),
v_{ij} = virtual age following ith repair in the jth PM interval ($j = 1, ..., k$; $i = 1, ..., n_j$),
v_{0j} = virtual age following $(j-1)$th PM ($j = 2, ..., k$), with $v_{01} = 0$.

The system's virtual age at time t is therefore given by

$$v(t) = v_{i-1,j} + t - t_{i-1,j} \quad \text{for} \quad t_{i-1,j} \leqslant t < t_{ij} \quad (j = 1, ..., k;\ i = 1, ..., n_j). \tag{1}$$

If model I is used for both types of maintenance action, then

$$\begin{aligned} v_{ij} &= v_{i-1,j} + \delta_{CM}(t_{ij} - t_{i-1,j}), \\ v_{0j} &= v_{0,j-1} + \delta_{PM}(v_{n_{j-1},j-1} - v_{0,j-1} + t_{0j} - t_{n_{j-1},j-1}), \end{aligned} \tag{2}$$

while, for model II, we have

$$\begin{aligned} v_{ij} &= \delta_{CM}(v_{i-1,j} + t_{ij} - t_{i-1,j}), \\ v_{0j} &= \delta_{PM}(v_{n_{j-1},j-1} + t_{0j} - t_{n_{j-1},j-1}). \end{aligned} \tag{3}$$

Note that, in the case of minimal CM ($\delta_{CM} = 1$), result (1) simplifies to

$$v(t) = t - (t_{0j} - v_{0j}) \quad \text{for} \quad t_{0j} \leqslant t < t_{0,j+1} \quad (j = 1, ..., k),$$

where $v_{0j} = \delta_{PM} t_{0j}$ for model I and $v_{0j} = \sum_{i=1}^{j-1} \delta_{PM}^{i}(t_{0,j-i+1} - t_{0,j-i})$ for model II.

In the case of perfect PM ($\delta_{PM} = 0$), result (1) holds with $v_{0j} = 0$ and $v_{ij} = \delta_{CM}(t_{ij} - t_{0j})$ for model I and $v_{ij} = \sum_{l=1}^{i} \delta_{CM}^{l}(t_{i-l+1,j} - t_{i-l,j})$ for model II.

3. Point process model for system failures

From Lawless & Thiagarajah (1996), the conditional (or complete) intensity function for the point process of system failures is defined by

$$u(t;\,H_t) = \lim_{\Delta t \downarrow 0} \frac{\Pr\{\text{a failure in } (t, t + \Delta t) | H_t\}}{\Delta t}, \tag{4}$$

where H_t is the history of the failure process up to time t.

Assuming no more than one failure can occur in time Δt, it follows that $u(t;\,H_t)\Delta t$ gives the expected number of failures occurring in $(t, t + \Delta t)$ with $u(t;\,H_t)$ the rate of occurrence of failures at t *given the previous history of failures up to* t.

Now, the chance of a failure at time t in both our imperfect-maintenance models depends only on the system's virtual age. Hence, using result (1),

$$u(t;\,H_t) = r[v(t)] = r(v_{i-1,j} + t - t_{i-1,j}) \quad \text{for } t_{i-1,j} \leqslant t < t_{ij}, \tag{5}$$

where $r(x)$ is the hazard rate function for the time to first system failure.

If $N(t)$ denotes the number of failures occurring up to time t, then $M(t) = \mathrm{E}\{N(t)\}$ is termed the *expected failure count function* and $m(t) = M'(t) = \lim_{\Delta t \downarrow 0} \frac{\Pr\{\text{a failure in } (t, t+\Delta t)\}}{\Delta t}$ the *rate of occurrence of failures at* t. For general δ_{CM} and δ_{PM}, $m(t)$ and $u(t;\,H_t)$ are distinct functions.

However, closed-form expressions for $M(t)$ exist in some special cases. For example, with periodic PMs at times jT $(j = 1, 2, \ldots)$ and minimal CM ($\delta_{\mathrm{CM}} = 1$), we have

$$M(t;\,T) = \sum_{i=1}^{j-1} [R(v_{0i}+T) - R(v_{0i})] + R(v_{0j}+t-(j-1)T) - R(v_{0j}) \quad \text{for } (j-1)T \leqslant t < jT, \tag{6}$$

where $v_{0i} = (i - 1)\delta_{\mathrm{PM}}T$ (Model I), $v_{0i} = \left(\frac{1-\delta_{\mathrm{PM}}^{i-1}}{1-\delta_{\mathrm{PM}}}\right)\delta_{\mathrm{PM}}T$ (Model II), and $R(x) = \int_0^x r(u)du$ is the corresponding cumulative hazard function. Note that the previous notation for the expected failure count function has now been extended to indicate the dependence on T.

Note also that, for perfect PM ($\delta_{\mathrm{PM}} = 0$), result (6) reduces to the simple form

$$M(t;\,T) = (j - 1)R(T) + R(t - (j - 1)T) \quad \text{for } (j - 1)T \leqslant t < jT. \tag{7}$$

For general δ_{CM} and δ_{PM}, $M(t;\,T)$ has to be estimated by simulation and this will be discussed in section 5.

EXAMPLE 1 Suppose that the time to first failure (in hours) has a Weibull distribution with $R(x) = (0.001x)^{2.5}$, $T = 1000$, $\delta_{\mathrm{CM}} = 1$, and $\delta_{\mathrm{PM}} = 0.2$. Figure 1 shows the graphs of the function $M(t;\,T)$ given in result (6) for model I and model II. Notice that, as t increases, the expected number of system failures for model II is smaller.

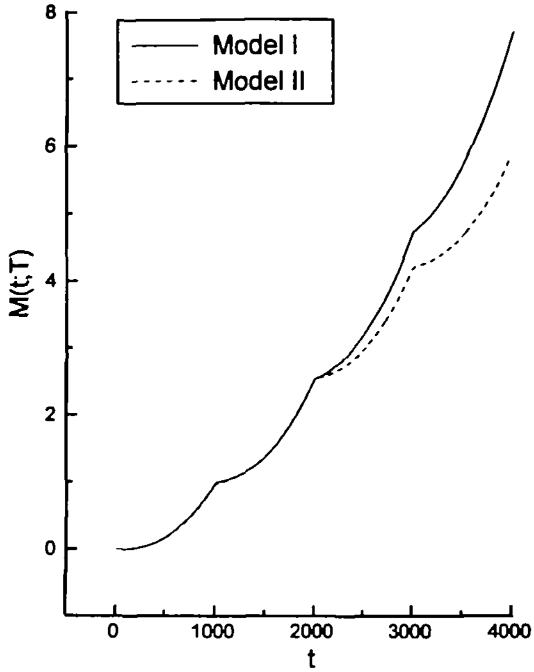

FIG. 1. Expected failure count functions under minimal CM

4. Parameter estimation

If observation of the system's behaviour ceases at failure time $t_{n_k,k}$ then, using Anderson *et al.* (1993), the likelihood function is given by

$$L(\theta_1, \ldots, \theta_m; \delta_{\mathrm{CM}}, \delta_{\mathrm{PM}}) = \prod_{j=1}^{k} \prod_{i=1}^{n_j} u(t_{ij}; H_{t_{ij}}) \exp\left\{ - \int_0^{t_{n_k,k}} u(t; H_t) \mathrm{d}t \right\}, \tag{8}$$

where $\theta_1, \ldots, \theta_m$ are the parameters of the distribution of X_{11}, the time to first system failure.

Using results (1) and (5), and assuming that X_{11} is Weibull with $r(x) = \lambda\beta(\lambda x)^{\beta-1}$ and $R(x) = (\lambda x)^{\beta}$, where $\lambda > 0$ and $\beta > 0$, it follows that the corresponding log-likelihood is

$$\begin{aligned} \ell(\lambda, \beta; \delta_{\mathrm{CM}}, \delta_{\mathrm{PM}}) = & \sum_{j=1}^{k} \sum_{i=1}^{n_j} \ln\, r(v_{i-1,j} + t_{ij} - t_{i-1,j}) \\ & - \sum_{j=1}^{k-1} \sum_{i=1}^{n_j+1} \int_{t_{i-1,j}}^{t_{ij}} r(v_{i-1,j} + t - t_{i-1,j})\, \mathrm{d}t \\ & - \sum_{i=1}^{n_k} \int_{t_{i-1,k}}^{t_{ik}} r(v_{i-1,j} + t - t_{i-1,j}) \mathrm{d}t \end{aligned}$$

$$= (\beta \ln \lambda + \ln \beta) \sum_{j=1}^{k} n_j + (\beta - 1) \sum_{j=1}^{k} \sum_{i=1}^{n_j} \ln(v_{i-1,j} + t_{ij} - t_{i-1,j})$$

$$-\lambda^{\beta} \left\{ \sum_{j=1}^{k-1} \sum_{i=1}^{n_j+1} \left[(v_{i-1,j} + t_{ij} - t_{i-1,j})^{\beta} - v_{i-1,j}^{\beta} \right] \right.$$

$$\left. + \sum_{i=1}^{n_k} \left[(v_{i-1,k} + t_{ik} - t_{i-1,k})^{\beta} - v_{i-1,k}^{\beta} \right] \right\}. \quad (9)$$

Note that, if a number of identical systems are observed and the CM and PM times recorded, then the corresponding log-likelihood is just a sum of functions of the form given in result (9). This is the case in the example below.

The maximum-likelihood estimates (MLEs) $\hat{\lambda}$, $\hat{\beta}$, $\hat{\delta}_{CM}$, and $\hat{\delta}_{PM}$ can be found by direct maximization of ℓ (or, equivalently, by minimizing $-\ell$) using the Nelder–Mead simplex algorithm (Press *et al.* (1992)) implemented in NAG routine E04CCF. Before the method is used, however, the transformations

$$\lambda' = \ln \lambda, \quad \beta' = \ln \beta, \quad \delta'_{CM} = \ln \left(\frac{\delta_{CM}}{1 - \delta_{CM}} \right), \quad \delta'_{PM} = \ln \left(\frac{\delta_{PM}}{1 - \delta_{PM}} \right)$$

need to be made to ensure that all parameters have range $(-\infty, \infty)$.

Provided the number of observed CM times and PM times is large, the likelihood-ratio test statistic for each parameter is approximately χ^2-distributed with 1 degree of freedom. This provides an efficient method for obtaining confidence intervals (Doganaksoy & Schmee 1993).

The likelihood ratio statistic for λ is defined by

$$W_1(\lambda) = -2 \left[\ell(\lambda, \tilde{\beta}; \tilde{\delta}_{CM}, \tilde{\delta}_{PM}) - \ell(\hat{\lambda}, \hat{\beta}; \hat{\delta}_{CM}, \hat{\delta}_{PM}) \right], \quad (10)$$

where $\tilde{\beta}, \tilde{\delta}_{CM}$, and $\tilde{\delta}_{PM}$ are the MLEs of β, δ_{CM}, and δ_{PM} for a fixed value of λ. A 100 $(1 - \alpha)$% confidence interval for λ therefore consists of the set of values of λ for which $W_1(\lambda) \leqslant \chi^2(1; 1 - \alpha)$, the $1 - \alpha$ quantile of the χ^2 distribution with 1 df. The corresponding lower and upper likelihood-ratio (LR) confidence limits for λ are the two values λ_L and λ_U that satisfy

$$\ell(\lambda, \tilde{\beta}; \tilde{\delta}_{CM}, \tilde{\delta}_{PM}) = \ell(\hat{\lambda}, \hat{\beta}, \hat{\delta}_{CM}, \hat{\delta}_{PM}) - \tfrac{1}{2}\chi^2(1; 1 - \alpha). \quad (11)$$

The left-hand side of equation (11) gives the profile log-likelihood function for λ and the right-hand side is a constant. λ_L and λ_U are found by successive evaluation of this function until the equation is satisfied. Similar results to (10) and (11) apply to obtain 100 $(1 - \alpha)$% LR confidence limits for β, δ_{CM}, and δ_{PM}.

EXAMPLE 2 We considered Baker's (1991) CM and PM data from medical equipment (syringe-driver infusion pumps) used in a large teaching hospital. A sample of 9 of the pumps produced a total of 80 CM times and 30 PM times. Jack (1997) used this same data assuming minimal CM and imperfect PM. The following parameter estimates, 95% confidence limits and maximized log-likelihood were obtained using Model I for both CMs and PMs

$$\hat{\lambda} = 0.000958, \quad \lambda_L = 0.000811, \quad \lambda_U = 0.00120,$$
$$\hat{\beta} = 2.71, \quad \beta_L = 2.05, \quad \beta_U = 3.45,$$
$$\hat{\delta}_{CM} = 1.000, \quad \delta_{CM}^L = 0.953, \quad \delta_{CM}^U = 1.000,$$
$$\hat{\delta}_{PM} = 0.512, \quad \delta_{PM}^L = 0.361, \quad \delta_{PM}^U = 0.759,$$
$$\ell(\hat{\lambda}, \hat{\beta}, \hat{\delta}_{CM}, \hat{\delta}_{PM}) = -524.06.$$

The corresponding values using Model II for both types of maintenance were:

$$\hat{\lambda} = 0.000946, \quad \lambda_L = 0.000786, \quad \lambda_U = 0.00121,$$
$$\hat{\beta} = 2.48, \quad \beta_L = 1.87, \quad \beta_U = 3.18,$$
$$\hat{\delta}_{CM} = 1.000, \quad \delta_{CM}^L = 0.896, \quad \delta_{CM}^U = 1.000,$$
$$\hat{\delta}_{PM} = 0.789, \quad \delta_{PM}^L = 0.643, \quad \delta_{PM}^U = 0.987,$$
$$\ell(\hat{\lambda}, \hat{\beta}, \hat{\delta}_{CM}, \hat{\delta}_{PM}) = -526.01.$$

These results validate Jack's (1997) assumption of minimal CM for this equipment.

5. Simulating system behaviour

Let $X_{ij} = T_{ij} - T_{i-1,j}$ be the random variable representing the time between the $(i-1)$th and the ith failure in the jth PM interval. Suppose that X_{11} (the time to first system failure) has survivor function $\bar{F}(x) = \Pr(X_{11} > x)$, hazard rate function $r(x) = -\bar{F}'(x)/\bar{F}(x)$, and cumulative hazard function $R(x) = \int_0^x r(u)\mathrm{d}u$. It follows that X_{ij} has conditional survivor function

$$\Pr\{X_{ij} > x | v_{i-1,j}\} = \frac{\bar{F}(v_{i-1,j} + x)}{\bar{F}(v_{i-1,j})} = \exp\{-[R(v_{i-1,j} + x) - R(v_{i-1,j})]\}. \quad (12)$$

If u_{ij} denotes a uniform (0,1) variate, then the inversion method for random variate generation (Dagpunar 1988) gives

$$\exp\{-[R(v_{i-1,j} + x) - R(v_{i-1,j})]\} = u_{ij},$$

or

$$x = -v_{i-1,j} + R^{-1}[R(v_{i-1,j}) - \ln u_{ij}].$$

The required generator for system failure times $T_{ij} = T_{i-1,j} + x$ is therefore

$$T_{ij} = T_{i-1,j} - v_{i-1,j} + R^{-1}[R(v_{i-1,j}) - \ln u_{ij}] \quad (j = 1, ..., k;\ i = 1, ..., n_j). \quad (13)$$

Assuming periodic PMs with period T, it follows that n_j is the largest i such that $T_{ij} \leqslant jT$.

Now, if X_{11} is Weibull with $\bar{F}(x) = \exp\{-(\lambda x)^\beta\}$, then $R(x) = (\lambda x)^\beta$ is easily invertible, and it is straightforward to compute the system failure times from result (13). $M(t;\ T)$,

TABLE 1
Optimal values of T and k for model I

δ_{CM}	T^*	k^*	$C(T^*, k^*)$	% increase
1.0	700	5	0.0669	
0.9	700	5	0.0656	0.00
0.8	700	6	0.0643	0.31
0.7	700	6	0.0629	1.11
0.6	875	5	0.0611	2.45
0.5	1225	4	0.0592	4.22

the expected failure count function for the system, can be estimated by performing a large number of independent simulations (10000, say) of the failure process and then computing the average number of failures occurring up to time t for each $t \geqslant 0$. The accuracy of the estimate, $\hat{M}(t; T)$, was tested against the known forms of the function given in results (6) and (7), producing agreement to three significant figures over an interval of 5 PM cycles with $R(x) = (0.001x)^{2.5}$ and $T = 1000$.

The use of this simulation estimate to determine optimal PM and replacement intervals for the system will now be discussed.

6. Optimal preventive maintenance and replacement

Suppose the system is subjected to periodic PM at times jT ($j = 1, ..., k - 1$), and then replaced by a new one after time kT, and let c_0, c_1, and c_2 denote the costs of a replacement, a PM, and a CM, respectively. Using result (6), i.e. assuming minimal CM, Lui *et al.* (1995) computed optimal values of T and K by minimizing the long-run expected total cost per unit time:

$$C(T, k) = \frac{c_0 + (k-1)c_1 + c_2 M(kT; T)}{kT} \tag{14}$$

The simulation estimate $\hat{M}(kT; T)$ discussed in section 5 can be used to compute the corresponding optimal values for the case of general δ_{CM}.

EXAMPLE 3 Suppose that X_{11} (in hours) is Weibull with $R(x) = (0.001x)^{2.5}$ ($E(X_{11}) = 887$ hours) and PMs can only be carried out at multiples of $T_0 = 175$ hours ($\Rightarrow T = nT_0$ ($n = 1, 2, ...$) is discrete). Let $c_0 = 100$, $c_1 = 10$, $c_2 = 20$, and $\delta_{PM} = 0.2$. Table 1 shows the optimal maintenance policy according to model I when δ_{CM} varies from 1 down to 0.5. The values of the two discrete decision variables were found using a simple numerical search procedure. The last column in the table shows the percentage increase in cost rate when the minimal CM ($\delta_{CM} = 1$) policy, $T = 700$ and $k = 5$, is used instead of the appropriate optimal policy. It can be seen that δ_{CM} has to be much less than 1 before the effect of using the incorrect policy is significant.

7. Conclusions

This is the first paper to model the effect of both imperfect CM and imperfect PM using the concept of reduction in virtual or effective age. Simulation, optimization, and statistical models are all developed. A simulation algorithm is given which generates system failure times under a given periodic PM strategy. The expected number of failures the system will experience up to any given time can then be estimated, and optimal PM and replacement times found. However, this optimization procedure relies on the availability of parameter estimates for the system's lifetime distribution and the age reduction factors for both CM and PM. Statistical techniques are given to obtain these estimates and their confidence intervals from given maintenance data.

REFERENCES

ANDERSON, P. K., BORGAN, O., GILL, R. D., & KEIDING, N. 1993 Statistical methods based on counting processes. Springer-Verlag, New York.

BAKER, R. D. 1991 Testing the efficacy of preventive maintenance. *Journal of the Operational Research Society* **42**, 493–503.

DAGPUNAR, J. S. 1988 Principles of random variate generation. Oxford University Press, Oxford.

DOGANAKSOY, N., & SCHMEE, J. 1993 Comparisons of approximate confidence intervals for distributions used in life data analysis. Technometrics **35**, 175–184.

JACK, N. 1997 Analysing event data from a repairable machine subject to imperfect preventive maintenance. *Quality and Reliability Engineering International* **13**, 183–186.

KIJIMA, M., MORIMURA, H., & SUZUKI, Y. 1988 Periodical replacement problem without assuming minimal repair. *European Journal of Operational Research* **37**, 194–203.

LAWLESS, J. F., & THIAGARAJAH, K. 1996 A point process model incorporating renewals and time trends, with application to repairable systems. *Technometrics*, **38**, 131–138.

LIU, X.-G., MAKIS, V., & JARDINE, A. K. S. 1995 A replacement model with overhauls and repairs. *Naval Research Logistics* **42**, 1063–1079.

PRESS, W. H., TEUKOLSKY, S. A., KETTERLING, W. T. & FLANNERY, B. P. 1992 Numerical Recipies in Fortran: The Art of Scientific Computing, 2nd edition, Cambridge University Press, Cambridge.