ELSEVIER

Reliability Engineering and System Safety 70 (2000) 305–312

RELIABILITY ENGINEERING & SYSTEM SAFETY

www.elsevier.com/locate/ress

A new model with bathtub-shaped failure rate using an additive Burr XII distribution

F.K. Wang*

Department of Business Administration, Chang Gung University, No. 259, Wen-Hwa 1 Road, Kwei-Shan, Tao-Yuan 333, Taiwan

Received 12 July 1999; accepted 6 June 2000

Abstract

It is a common situation that the failure rate function has a bathtub shape for many mechanical and electronic components. A simple model based on adding two Burr XII distributions is presented for modeling this type data. The graphical estimation on probability paper is illustrated, and examples of its usage are presented. The Akaike Information Criterion was used for judging the adequacy of the models presented for the numerical examples. It can be seen that the proposed model is a very competitive model for describing the bathtub-shaped failure rate lifetime data. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Bathtub-shaped failure rate; Burr XII distribution; Lifetime data

1. Introduction

The probability distribution of the time-to-failure of a device can be characterized by the failure rate, $h(t) = f(t)/R(t)$, where $f(t)$ denotes the density function and $R(t)$ the reliability function which is also called the survivorship function. The quantity $h(t)$ represents the probability that a device of age t will fail in the small interval of time t to $t + \mathrm{d}t$, which is also called the hazard function. It is a common situation that the failure rate function has a bathtub shape for many mechanical and electronic components. Models which allow only monotone failure rates might not be appropriate or adequate for modeling the whole bathtub-shaped data.

Several models have been proposed to model the bathtub-shaped failure rates; see Refs. [1–5] and they are summarized in Table 1.

Among them, a variety of methods for estimation and testing based on general principles such as method of moments, least squares, and maximum likelihood have been examined and discussed for these models. However, most models are not practical to be used by reliability engineers. In order to identify the type of hazard rate of a lifetime data, many approaches have been proposed. Glaser [6] has obtained sufficient conditions to ensure whether a lifetime model has a bathtub failure rate or not. He also reported that the corresponding mixture based on Weibull densities is not bathtub-shaped. In this study, a graphical method based on total time on test (TTT) transform introduced by Barlow and Campo [7] and further extended by Bergman and Klefsjo [8] will be used to illustrate the variety of hazard-rate shapes. It has been shown that the hazard function of $F(t)$ increases (decreases) if the scaled TTT-transform, $\phi_F(t) = H_F^{-1}(t)/H_F^{-1}(1)$, where $H_F^{-1}(t) = \int_0^{F^{-1}(t)} R(u)\mathrm{d}u$, $0 \le u \le 1$, is concave (convex). In addition, for a distribution with bathtub (unimodal) failure rate the TTT-transform is first convex (concave) and then concave (convex).

The parameters of the model by Haupt and Schabe [3] and the additive Weibull model by Xie and Lai [5] can be estimated by the probability plotting techniques. It is the intention to study another practical model for the bathtub-shaped failure rate function. Burr (1942) constructed the Burr system of distributions for the expressed purposed of fitting the cumulative density function to a diversity of frequency data forms. The Burr has already been used in a variety of non-reliability applications such as quality control, acceptance sampling and medical study. Zimmer et al. [9] discussed the Burr XII distribution in reliability analysis and provided a useful model for representing failure data. This model is very flexible in modeling various types of lifetime distribution. The reliability and the failure rate function of the three-parameter model are given by

$$R(t) = \frac{1}{(1 + (t/s)^c)^k}, \qquad t, s, c, k > 0 \tag{1}$$

$$h(t) = \frac{kc(t/s)^{c-1}}{s(1 + (t/s)^c)}, \qquad t, s, c, k > 0 \tag{2}$$

* Tel.: +886-3-328-3016, ext. 5408; fax: +886-3-327-1304.
E-mail address: fukwun@mail.cgu.edu.tw (F.K. Wang).

PII: S0951-8320(00)00066-1

Table 1
Some models for a bathtub-shaped failure rate

Author	Reliability function and hazard function	Characteristics
Hjorth [1]	$R(t) = 1 - F(t) = \frac{e^{-\delta t^2/2}}{(1+\beta t)^{\theta/\beta}}, t \geq 0$ $h(t) = \delta t + \frac{\theta}{1+\beta t}$	$\theta = 0 \rightarrow$ Weibull distribution $\delta = \beta = 0 \rightarrow$ exponential distribution $\delta = 0 \rightarrow$ decreasing failure rate $\delta \geq \theta\beta \rightarrow$ increasing failure rate $0 < \delta < \theta\beta \rightarrow$ bathtub curve
Haupt and Schabe [3]	$R(t) = 1 + \beta - \sqrt{\beta^2 + (1+2\beta)t/t_0}\ 0 \leq t \leq t_0$ $h(t) = \frac{1+2\beta}{2t_0 A(1+\beta-A)}, 0 \leq t \leq t_0$ where $A = \sqrt{\beta^2 + (1+2\beta)t/t_0}$	$\beta \leq -1/3 \rightarrow$ increasing failure rate $\beta \geq 1 \rightarrow$ increasing failure rate $-1/3 < \beta < 1 \rightarrow$ bathtub curve
Mudholkar and Srirastava [4]	$R(t) = 1 - [1 - \exp(-t/\sigma)^{\alpha}]^{\theta}, t \geq 0$ $h(t) = \frac{\alpha\theta(1-A)^{\theta-1}A(t/\sigma)^{\alpha-1}}{\sigma[-(1-A)^{\theta}]}, t \geq 0$ where $A = \exp(-t/\sigma)^{\alpha}$	$\theta = 1 \rightarrow$ Weibull distribution $\alpha = \theta = 0 \rightarrow$ exponential distribution $\alpha, \theta < 1 \rightarrow$ decreasing failure rate $\alpha, \theta > 1 \rightarrow$ increasing failure rate $\alpha > 1, \theta < 1 \rightarrow$ bathtub curve or increasing $\alpha < 1, \theta > 1 \rightarrow$ unimodal or decreasing
Xie and Lai [5]	$R(t) = \exp(-(at)^b - (ct)^d), t \geq 0, b > 1, d < 1$ $h(t) = ab(at)^{b-1} + cd(ct)^{d-1}, t \geq 0$	Bathtub curve

It can be seen that when $c < 1$, the failure rate function decreases, and when $c > 2$, the failure rate function reaches a maximum, and then decreases. The ranges of values where the hazard increases can be manipulated using s. Thus, the Burr can represent an increasing, decreasing, unimodal, or essentially constant hazard rate in specified ranges. In this paper, we propose a simple model, which is based on adding two Burr XII distributions. The model is an additive one in the sense that the failure rate function is expressed as the sum of two failure rate functions of the Burr XII form. The graphical approach on probability paper is illustrated for parameter estimation. Some examples are used to demonstrate the applicability of this additive model.

2. The additive Burr XII model

The additive Burr XII model combines two Burr XII distributions; one has a decreasing failure rate and another has an increasing failure rate. Thus, the hazard function for the additive Burr XII is given by

$$h(t) = \frac{k_1 c_1 (t/s_1)^{c_1-1}}{s_1[1+(t/s_1)^{c_1}]} + \frac{k_2 c_2 (t/s_2)^{c_2-1}}{s_2[1+(t/s_2)^{c_2}]}, \tag{3}$$

$$t, k_1, k_2, s_1, s_2 \geq 0, 0 < c_1 < 1, c_2 > 2$$

Then, the cumulative hazard function can be obtained as

$$H(t) = \int_0^t h(u)\mathrm{d}u = k_1 \ln[1+(t/s_1)^{c_1}] + k_2 \ln[1+(t/s_2)^{c_2}] \tag{4}$$

Based on this form of cumulative hazard function, the reliability function is given by

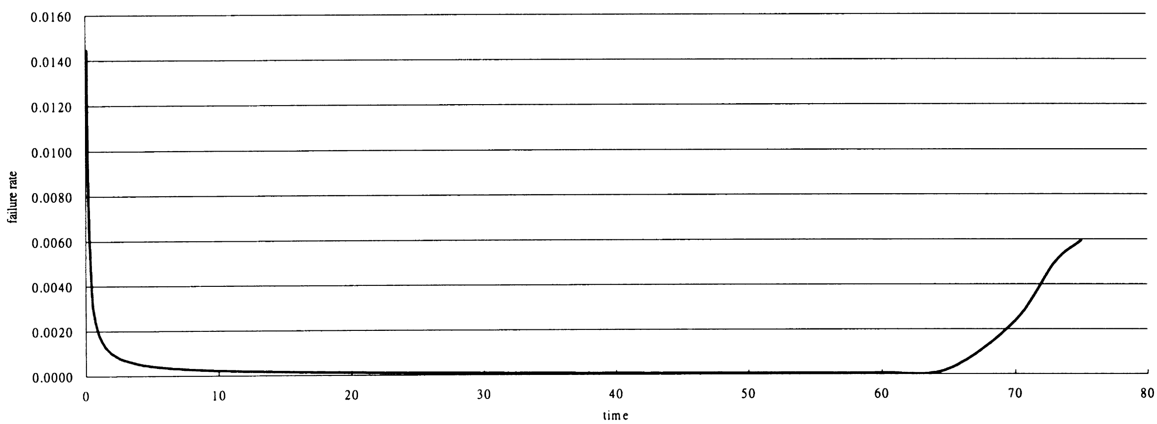
$$R(t) = \exp\{-H(t)\} = \exp\{-k_1 \ln[1+(t/s_1)^{c_1}] - k_2 \ln[1+(t/s_2)^{c_2}]\}, \quad t \geq 0 \tag{5}$$

The corresponding lifetime distribution function is then given by

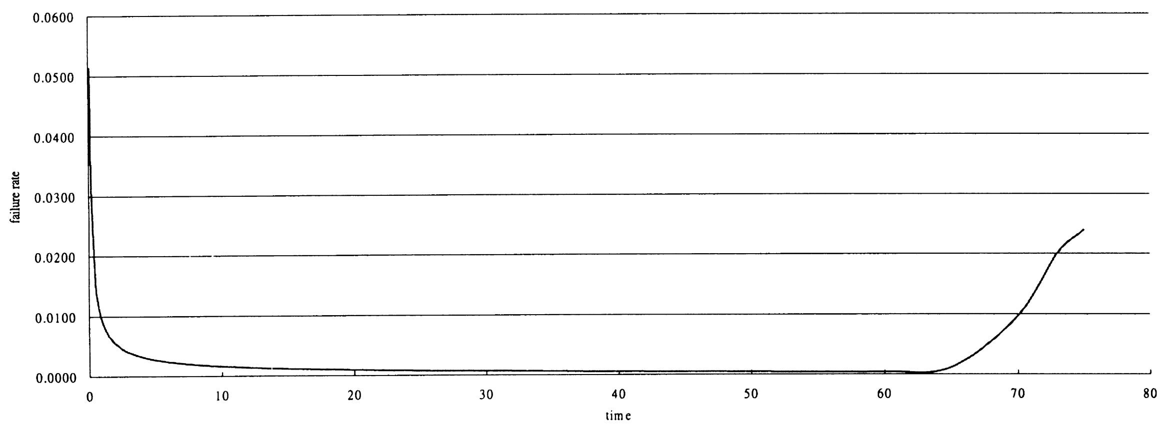
$$F(t) = 1 - R(t) = 1 - \exp\{-k_1 \ln[1+(t/s_1)^{c_1}] - k_2 \ln[1+(t/s_2)^{c_2}]\}, \quad t \geq 0 \tag{6}$$

Some plots using the hazard function are displayed in Fig. 1. It can be seen that this failure function has a bathtub-shaped curve.

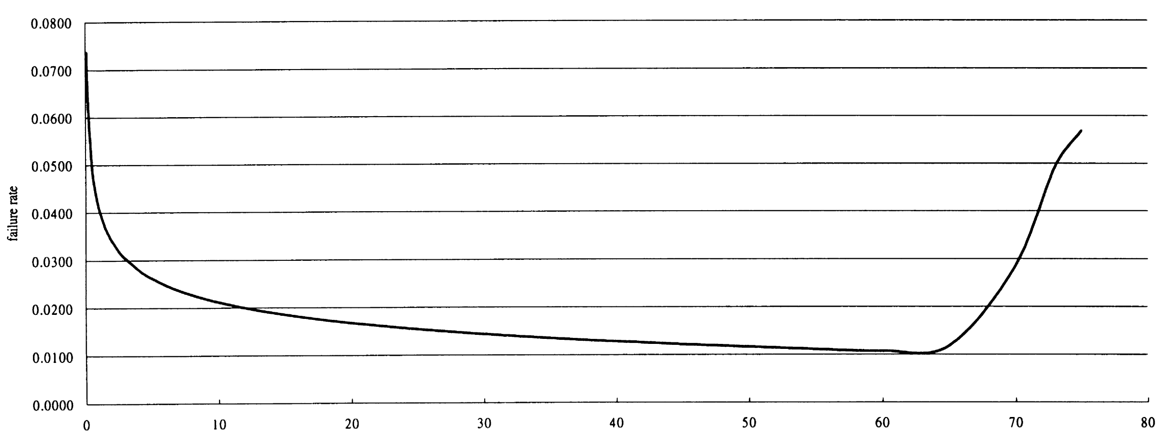
The physical background of this model is clear. A component fails because of the occurrence of a failure mode and usually there are different failure modes associated with a component. Each failure mode affects the component in a different way. Suppose that the component is affected by two major failure modes. Thus, the above model represents that the component can fail due to either of these two failure



$$c_1 = 0.1, k_1 = 1, s_1 = 2.17 \times 10^{17}, c_2 = 40, k_2 = 0.5, s_2 = 71.29$$



$$c_1 = 0.25, k_1 = 2, s_1 = 8.6 \times 10^{6}, c_2 = 40, k_2 = 2, s_2 = 71.29$$



$$c_1 = 0.75, k_1 = 3, s_1 = 204.88, c_2 = 40, k_2 = 4, s_2 = 71.29$$

Fig. 1. Some typical bathtub-shaped failure rate function using different values of the parameters.

modes. For a component that has a bathtub-shaped failure rate, the initial failures are usually caused by design faults and initial problems, which lead to a decreasing failure rate. Material fatigue or component aging usually causes the last part of the bathtub-shaped failure rate and this corresponds to an increasing failure rate. This model incorporates both types of failures and it can be used to analyze this kind of failure data collected without knowing what types of failure have occurred.

One of the uses of a bathtub curve is that we can determine the optimum burn-in time in the case when the initial failure rates is too high for the product to be released directly after production. Also, after a certain time, the product enters the wear-out phase and replacement should be considered. The decision can easily be made based on the model. For example, the product can only be released after burn-in when the failure rate is less than h_b to meet customers' requirement, then the optimum burn-in time can be determined by $h(t) = h_b$. Similarly, if the product has to be replaced by a new one when the failure rate is too high, higher than h_r, then the optimum replacement time can be determined by solving the following equation $h(t) = h_r$. Both equations can be solved numerically using standard algorithms. In addition, the hazard plot can be used to determine the optimum burn-in time or the optimum replacement time.

3. Probability plotting procedure

Probability plotting is a graphical method used to investigate whether an assumed model adequately fits a set of data. It helps the analyst to assess how well a given theoretical distribution fits the data and to estimate distribution parameters through least squares (LS) procedure. Plotting the data values against the corresponding estimated quantile values, where the scale is adjusted so that the relationship is linear for a given theoretical distribution, produces a probability plot. Thus, a linear pattern of points indicates agreement between the data distribution and the theoretical distribution; a non-linear pattern indicates that the assumed distribution is not a reasonable representation of the data. If one can take the log-transformation of both sides in Eq. (1), the Burr quantile function becomes

$$\ln([R(t_p)]^{-1/k} - 1) = c\ln(t_p) - c\ln(s) \tag{7}$$

Thus, the empirical value of $\ln([R(t_p)]^{-1/k} - 1)$ is plotted versus the ordered log-transformed data values. Here, $R(t_p)$ is estimated by Herd–Johnson procedure, that is

$$\hat{R}(t_p) = \prod_{r=1}^{f} \frac{n-r+1}{n-r+2}$$

where n is the number of units on test, r the order of each failure after order numbers have been assigned to all units based on their running times, and f the observed number of failures. In the case of complete data, the Herd–Johnson estimate is identical to the maximum likelihood (ML) estimation order-statistic, $i/(n+1)$. If k is unknown, it may be necessary to construct plots for a series of k values to obtain a linear pattern. If the plot when $k = 1$ is concave down, then smaller values for k tend to linearize the plot; if the plot when $k = 1$ is concave up, then larger values of k tend to linearize the plot. It is because of this concave–convex relationship of the plots in terms of k that we recommend that the analyst begin by fixing k first and then uses the linear plots to estimate c and s. Such estimates are easily obtained using spreadsheets. Since the model is based on two Burr XII distributions, the parameter estimation can be obtained by the following steps.

Step 1. Using the first few ordered lifetime data, plot the empirical value of $\ln([R(t_p)]^{-1/k} - 1)$ versus the ordered log-transformed data values to find the best fitting straight line, using LS, therefore, the parameter k_1 can be obtained. Then use the linear plot to estimate c_1 and s_1.

Step 2. Using the last few ordered lifetime data, plot the empirical value of $\ln([R(t_p)]^{-1/k} - 1)$ versus the ordered log-transformed data values to find the best fitting straight line, using LS, therefore, the parameter k_2 can be obtained. Then use the linear plot to estimate c_2 and s_2.

4. Examples

In order to demonstrate the proposed methodology, one example from literature and one real-world data are used. Through the use of graphical displays, the intent is that the reader can gain a perspective of the various meanings and associated interpretations. In addition, the Akaike Information Criterion (AIC) by Akaike [10] can be used to select the best model among several models. The AIC is given as follows:

$$\begin{aligned} \mathrm{AIC} = {} & -2 \times \log(\text{maximum log likelihood}) \\ & + 2 \times (\text{number of parameters fitted}) \end{aligned} \tag{8}$$

The best model for the data as determined by the AIC is the model with the lowest AIC value. The log likelihood function from n independent subjects in which t_i and c_i represent the failure time data and the censoring times of the ith unit, respectively, $i = 1, 2, \dots, n$, can be found in Cox and Oakes [11]. Then this function is given by

$$\ell(\Theta; t) = \sum_{\text{uncensored observations}} \log h(x_i; \Theta) + \sum \log R(x_i; \Theta) \tag{9}$$

where $x_i = \min(t_i, c_i)$ and Θ is the set of parameters of the model.

Example 1. Table 2 contains the times to failure of 50 devices by Aarset [12]; the TTT plot indicates a bathtub-shaped hazard rate in Fig. 2. Let us denote the failure times by $t_1, t_2, \dots, t_{50}$ and assume that $t_1 < t_2 < \dots < t_{50}$.

Table 2
Lifetimes of 50 devices (Aarset [10])

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18	18	18	21	32	36
40	45	46	47	50	55	60	63	63	67	67	67	67	72	75	79	82	82	83	84	84
84	85	85	85	85	85	86	86													

Table 3
The estimated parameters and AIC values in Table 2

Model	Estimated parameters	AIC	Rank
The exponentiated Weibull	$\alpha = 4.69$, $\theta = 0.146$ and $\sigma = 91.023$	748.41	4
Haupt and Schabe	$t_0 = 128.179$ and $\beta = 0.09$	470.52	2
The additive Weibull	$a \approx 0$, $b = 30.069$, $c = 0.0912$ and $d = 0.4996$,	532.89	3
The additive Burr XII	$c_1 = 0.5067$, $s_1 = 2137.215$, $k_1 = 5.5$, $c_2 = 152.93$, $s_2 = 85.2526$ and $k_2 = 0.5$,	444.63	1

Muldholkar and Srivastava [4] analyzed this data using an exponentiated-Wiebull model. The parameter estimation by maximum likelihood method is obtained by $\alpha = 4.69$, $\theta = 0.146$ and $\sigma = 91.023$. The mean time to failure is 38.4. Using the model by Haupt and Schabe [3], plotting $i/(n+1)$ versus $(n+1)t_i/i$, we can find an approximate a straight line with intercept $\alpha = 2\beta t_0/(1+2\beta) = 19.889$ and slope $\tan(\phi) = t_0/(1+2\beta) = 108.29$. This gives $t_0 = 128.179$ and $\beta = 0.09$. Using the additive-Weibull model by Xie and Lai [5], a Weibull plot based on the first fifteen points gives an estimate of the slope as 0.4996, which indeed corresponds to a decreasing failure rate at the beginning. The estimated slope for the last 10 points is 30.069, which corresponds to an increasing failure rate. The overall parameter estimation is given by $a \approx 0$, $b = 30.069$, $c = 0.0912$ and $d = 0.4996$. Using the proposed new model, a Burr plot based on the first 15 points gives an estimate of the slope as 0.5067 and the intercept as -3.885, which indeed corresponds to a decreasing failure rate at the beginning. The estimated slope for the last 10 points is 152.93 and the intercept as -674.39, which corresponds to an increasing failure rate. The overall parameter estimation is given by $c_1 = 0.5067$, $s_1 = 2137.215$, $k_1 = 5.5$, $c_2 = 152.93$, $s_2 = 85.2526$ and $k_2 = 0.5$. Furthermore, the estimated parameters and AIC values by several models are listed in

Table 4
Time to failure of 18 electronic devices

5	11	21	31	46	75	98	122	145	165	196	224	245	293	321	330	350	420

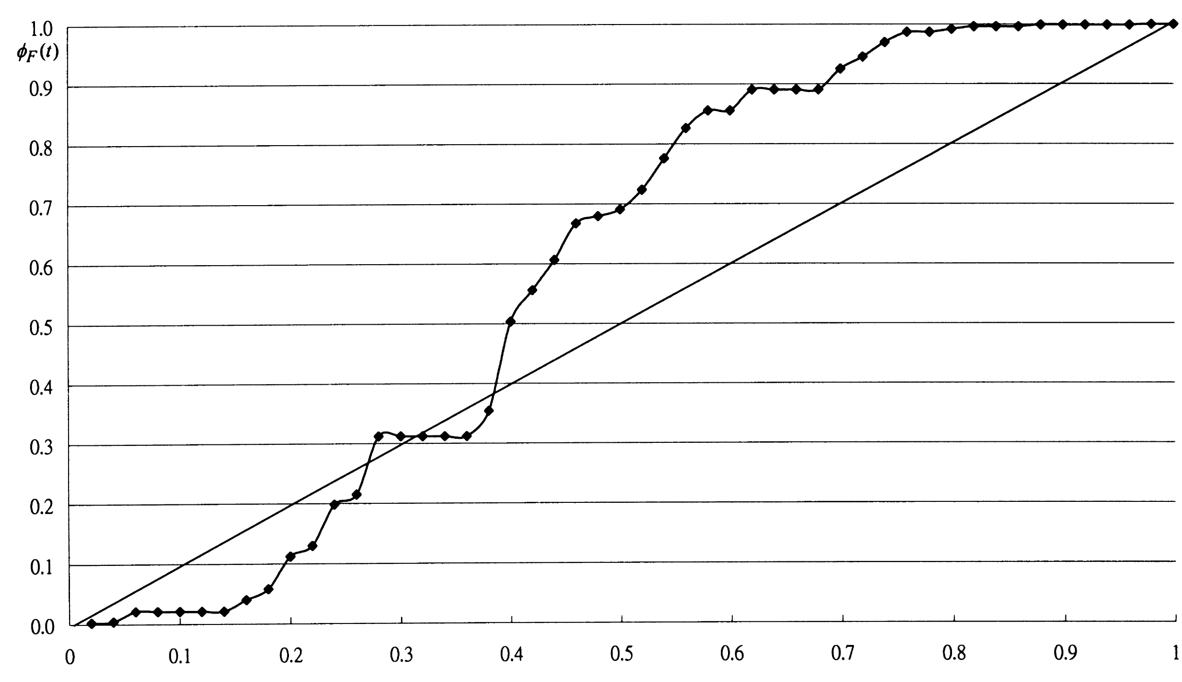


Fig. 2. TTT plot on the 50 observations in Table 2.

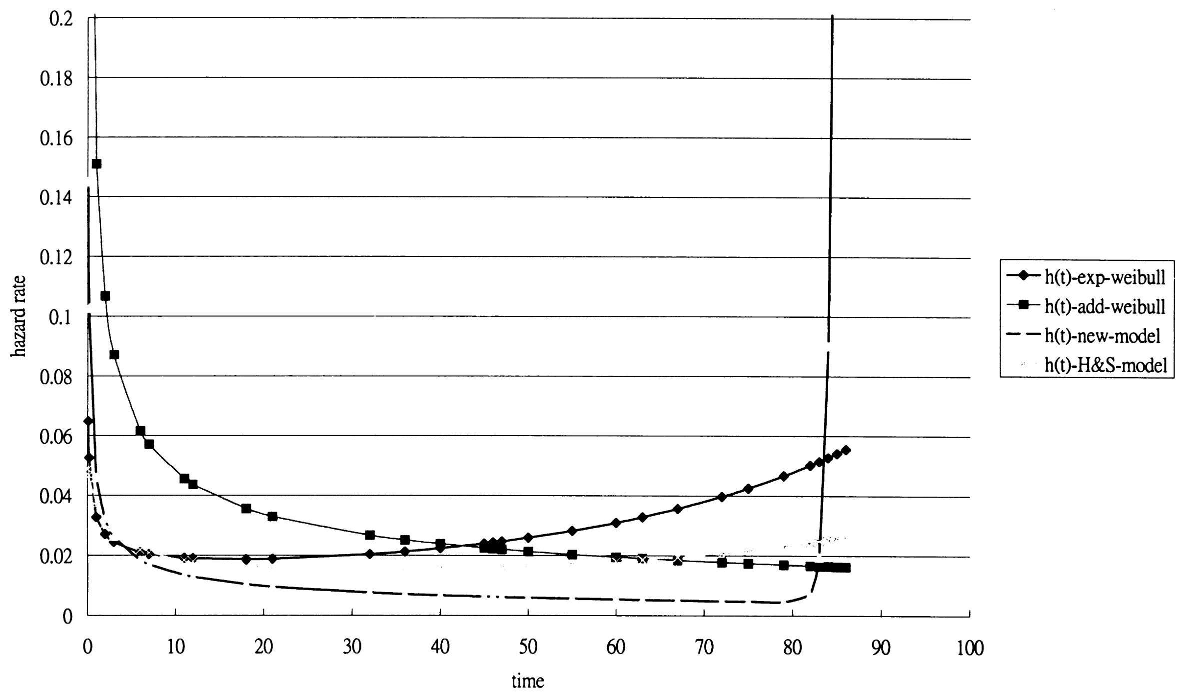


Fig. 3. The hazard function plot based on different models in Table 2.

Table 3. The result indicates that an additive Burr XII model has the lowest AIC value, and is the model chosen. The corresponding hazard plots by different models are given in Fig. 3. It can be seen that an additive Burr XII model is a very competitive model for describing the bathtub-shaped failure rate lifetime data.

Example 2. Table 4 represents the lifetime failure data of an electronic device. The TTT plot indicates a bathtub-shaped hazard rate in Fig. 4. Using the model by Haupt and Schabe [3], plotting $i/(n+1)$ versus $(n+1)t_i/i$, we can find an approximate straight line with intercept $\alpha = 2\beta t_0/(1+2\beta) = 91.105$ and slope $\tan(\phi) = t_0/(1+2\beta) = 389.94$. This gives $t_0 = 481.05$ and $\beta = 0.12$. Using the additive-Weibull model by Xie and Lai [5], a Weibull plot based on the

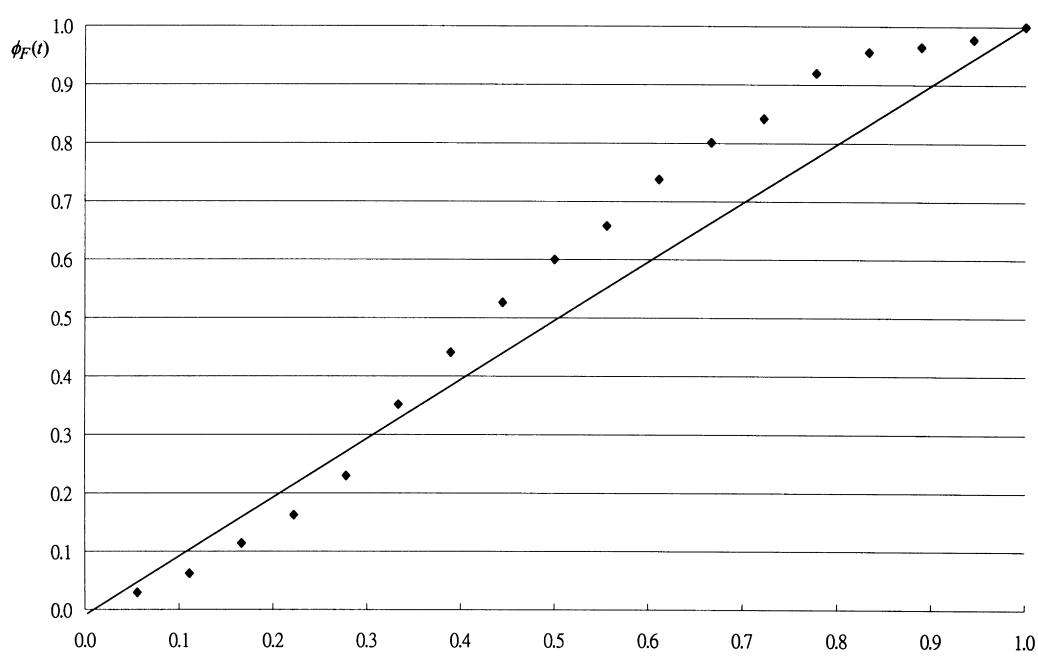


Fig. 4. TTT plot based on the 18 observations in Table 4.

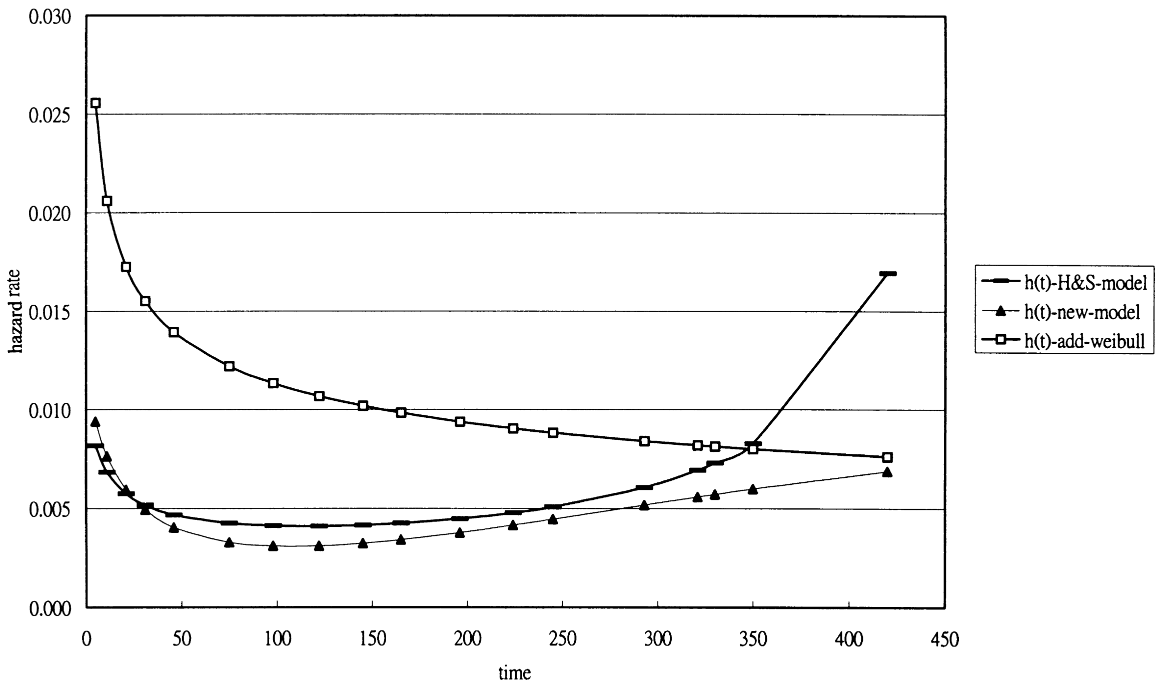


Fig. 5. The hazard function plot based on different models in Table 4.

first six points gives an estimate of the slope as 0.7266, which indeed corresponds to a decreasing failure rate at the beginning. The estimated slope for the last five points is 2.2589, which corresponds to an increasing failure rate. The overall parameter estimation is given by $a \approx 0$, $b = 2.2589$, $c = 0.0183$ and $d = 0.7266$. Using the proposed new model, a Burr plot based on the first six points gives an estimate of the slope as 0.961 and the intercept as -3.1024, which indeed corresponds to a decreasing failure rate at the beginning. The estimated slope for the last five points is 2.5012 and the intercept as -16.136, which corresponds to an increasing failure rate. The overall parameter estimation is given by $c_1 = 0.961$, $s_1 = 25.2368$, $k_1 = 0.28$, $c_2 = 2.5012$ $s_2 = 633.5275$ and $k_2 = 10$. Furthermore, the estimated parameters and AIC values by several models are listed in Table 5. The result indicates that an additive Burr XII model has the lowest AIC value, and is the model chosen. The corresponding hazard plots by different models are given in Fig. 5. It can be seen that the model by Xie and Lai [5] cannot be fitted with this data for describing the bathtub-shaped failure rate. However, the model by Haupt and Schabe [3] and the proposed model can describe the bathtub-shaped failure rate for this lifetime data.

Table 5
The estimated parameters and AIC values in Table 4

Model	Estimated parameters	AIC	Rank
Haupt and Schabe	$t_0 = 481.05$ and $\beta_1 = 0.12$,	221.77	2
The additive Weibull	$a \approx 0$, $b = 2.2589$, $c = 0.0183$ and $d = 0.7266$	247.05	3
The additive Burr XII	$c_1 = 0.961$, $s_1 = 25.2368$, $k_1 = 0.28$, $c_2 = 2.5012$, $s_2 = 633.5275$ and $k_2 = 10$	219.22	1

5. Conclusion

In this paper, an additive model based on the Burr XII distribution for lifetime data with bathtub-shaped failure rate was presented. The application of the model is straightforward. The parameter estimation can be estimated by the simple probability plot technique and easily obtained using spreadsheets. The results of this analysis showed that the proposed model has the lowest AIC value. The model can compute further studies such as MTTF, burn-in time and replacement time, when the parameters are estimated. However, the Maximum Likelihood Estimation (MLE) technique has several desirable properties for estimating the parameters of models. Using MLE to estimate the parameters, the optimization algorithms are often sensitive to the choice of starting values. The graphical approach in this paper can be used as the initial estimate.

Acknowledgements

The author wishes to gratefully acknowledge the referees of this paper who helped to clarify and improve the presentation. I am also grateful for partial support by National Science Council in Taiwan under the grant (NSC-88-2213-E182-005).

References

[1] Hjorth U. A reliability distribution with increasing, decreasing, constant and bathtub-shaped failure rates. Technometrics 1980;17:99–107.
[2] Rajarshi S, Rajarshi MB. Bathtub distribution: a review. Communication in Statistics: Theory and Method 1988;17:2597–621.
[3] Haupt E, Schabe H. A new model for a lifetime distribution with bathtub shaped failure rate. Mircoelectronic & Reliability 1992;32:633–9.
[4] Muldholkar GS, Srivastava DK. Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE Transactions on Reliability 1993;42:299–303.
[5] Xie M, Lai CD. Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. Reliability Engineering and Systems Safety 1995;52:87–93.
[6] Glaser RE. Bathtub and related failure rate characterizations. Journal of the American Statistical Association 1980;75:667–72.
[7] Barlow RE, Campo R. Total time on test processes and applications to failure data analysis. In: Barlow, Fussell, Singpurwalla, editors. Reliability and Fault Tree Analysis. SIAM, 1975. p. 235–72.
[8] Bergman B, Klefsjo B. A graphical method applicable to age replacement problems. IEEE Transactions on Reliability 1982;31:478–81.
[9] Zimmer WJ, Keats JB, Wang FK. The Burr XII distribution in reliability analysis. Journal of Quality Technology 1998;30:386–94.
[10] Akaike H. A new look at statistical model identification. IEEE Transactions on Automatic Control 1974;19:716–23.
[11] Cox DR, Oakes D. Analysis of survival data. London: Chapman & Hall, 1984.
[12] Aarset MV. How to identify a bathtub hazard rate. IEEE Transactions on Reliability 1987;36:106–8.