IMA Journal of Management Mathematics (2006) **17**, 235–243
doi:10.1093/imaman/dpi036
Advance Access publication on July 4, 2005

A novel repair model for imperfect maintenance

SHAOMIN WU† AND DEREK CLEMENTS-CROOME
School of Construction Management and Engineering, The University of Reading, Reading, RG6 6AW, UK

[Received on 4 March 2004; revised on 23 February 2005; accepted on 6 March 2005]

Commonly used repair rate models for repairable systems in the reliability literature are renewal processes, generalised renewal processes or non-homogeneous Poisson processes. In addition to these models, geometric processes (GP) are studied occasionally. The GP, however, can only model systems with monotonously changing (increasing, decreasing or constant) failure intensities. This paper deals with the reliability modelling of failure processes for repairable systems where the failure intensity shows a bathtub-type non-monotonic behaviour. A new stochastic process, i.e. an extended Poisson process, is introduced in this paper. Reliability indices are presented, and the parameters of the new process are estimated. Experimental results on a data set demonstrate the validity of the new process.

Keywords: renewal process; geometric process; corrective maintenance; preventive maintenance; maintenance policy.

1. Introduction

A repairable system is a system which, after failing to perform one or more of its functions satisfactorily, can be restored to fully satisfactory performance by any method, rather than the replacement of the entire system (Ascher & Feingold, 1984a). Repair models developed upon successive inter-failure times have been employed in many applications such as the optimisation of maintenance policies, decision making and whole-lifecycle cost analysis. With different repair levels, repair can be broken down into three categories (Yanez *et al.*, 2002): *perfect repair*, *normal repair* and *minimal repair*. A perfect repair can restore the system to as good as new, a normal repair is assumed to bring the system to any condition and a minimal repair (or imperfect repair) can restore the system to the state it was before failure. Examples of models for perfect, normal and minimal repair are renewal process (RP) models or homogeneous Poisson process (HPP) models, generalised renewal process and non-homogeneous Poisson process (NHPP) models, respectively. According to the dependence of failure intensities on time, repair models fall into three categories: *models with constant failure intensity* (e.g. HPP models), *models with time-dependent failure intensity* (e.g. NHPP models) and *models with repair time-dependent failure intensity* (e.g. geometric processes (GP) models, Lam, 1988).

Denote the survival time after the $(n-1)$th repair by X_n. $\{X_n,\ n = 1, 2, \ldots\}$ is assumed to be a sequence of independent exponential random variables in HPP models, a sequence of independent and identical random variables in RP models and a sequence of exponential random variables with time-dependent means in NHPP models. A GP (Lam, 1988) is a sequence of independent non-negative random variables, $\{X_n,\ n = 1, 2, \ldots\}$, such that $\{u^{n-1}X_n,\ n = 1, 2, \ldots\}$ (where $u(>0)$ is the parameter of the process) is an RP. In addition to the above-mentioned models, various repair models have been introduced under different assumptions (Block *et al.*, 1985; Brown & Proschan, 1983; Lindqvist

†Email: shaomin.wu@reading.ac.uk

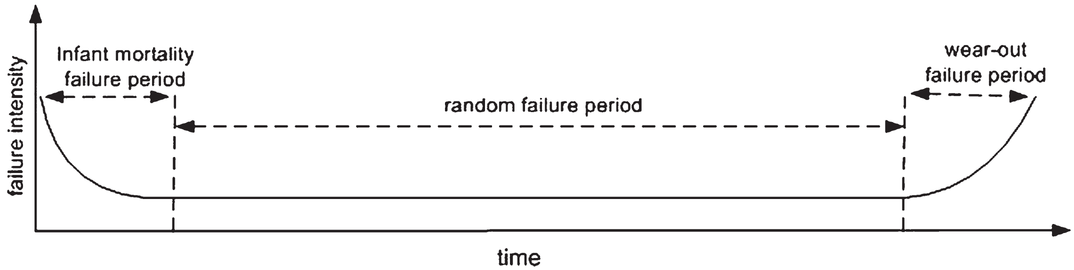

FIG. 1. Bathtub curve.

et al., 2003; Wang & Pham, 1996). The reader is referred to Pham & Wang (1996), Scarf (1997) and Wang (2002) for detailed and comprehensive discussions on the theoretic aspects and the applications of repair models.

A common assumption of both the perfect repair model and the minimal repair model is that a repaired system is exactly at a specified state, e.g. as good as new state. This assumption is too strict to be applied in some scenarios. The normal repair model, however, is more realistic because it assumes that a repaired system is either newer or older than that before its failure. In these models, the failure intensity of the system after repair is different from that before the failure occurs. Hence, it is reasonable to use a model such as a GP to deal with these situations as the failure intensity patterns after each repair can be differentiated in a GP.

A GP can describe a stochastically decreasing trend of the survival time of a system after repairs, or a stochastically increasing trend of the repair time after failures. It has interesting properties as they extend HPPs and RPs, and is easy to use. Researchers (Lam, 1988; Lam *et al.*, 2002; Lam & Zhang, 2003; Zhang, 1999, 2002; Wu *et al.*, 1994) made use of the GP to develop maintenance policies and analysed system reliability indices such as steady-state availability and system reliability. A weakness of the GP is that it can only model the system behaviour where the failure intensity of the system is monotonously increasing or decreasing with the operating time. It cannot model the system behaviour where the failure intensity of the system shows a non-monotonic or a more complicated trend. For example, a well-known curve for describing the change of failure intensity is bathtub-shaped. Patterns of failure intensities in the reliability theory and practice are usually illustrated using the bathtub curves which define the whole survival time of a population of devices. A typical bathtub curve (see Fig. 1) comprises three successive distinct periods: an infant mortality failure period with decreasing failure intensity, following a random failure period of time with a constant failure intensity and a wear-out failure period with an increasing failure intensity. The GP can only model system behaviour within one of the three periods in a bathtub curve, such as only the infant mortality failure period, only the random failure period or only the wear-out failure period.

In this paper, we introduce a new NHPP. The new process can model the system behaviour within the whole lifetime. That is, it can model the failure pattern from the infant mortality failure period, to the random failure period, to the wear-out failure period in a bathtub curve. The parameters of the new process are estimated, and a data set from Davis (1952) is borrowed to demonstrate the validity of the new process.

The paper is structured as follows. Section 2 introduces a new stochastic process, i.e. called an extended Poisson process (EPP). Section 3 derives the expected number of failures and the expected system reliability of the EPP within a given time interval. Section 4 offers the expected cost of a cycle under maintenance assumptions. Section 5 introduces an approach to estimating the parameters of the EPP and gives a numerical example for validating the new process. Section 6 presents concluding remarks and our further work.

2. Models

DEFINITION 1 (Ross, 1996)

(a) A random variable ξ is said to be stochastically not less (not greater) than another random variable ζ, denoted by $\xi \geqslant_{\mathrm{st}} \zeta$ ($\xi \leqslant_{\mathrm{st}} \zeta$) if $\Pr(\xi > w) \geqslant \Pr(\zeta > w)$ ($\Pr(\xi > w) \leqslant \Pr(\zeta > w)$) for all real w.

(b) A stochastic process $\{\xi_n,\ n = 1, 2, \ldots\}$ is said to be stochastically non-decreasing (non-increasing) if $\xi_n \leqslant_{\mathrm{st}} \xi_{n+1}$ ($\xi_n \geqslant_{\mathrm{st}} \xi_{n+1}$) for all $n = 1, 2, \ldots$. If $\xi_n <_{\mathrm{st}} \xi_{n+1}$ ($\xi_n >_{\mathrm{st}} \xi_{n+1}$), the stochastically increasing (decreasing) processes are defined.

DEFINITION 2 (Lam, 1988) A sequence of non-negative independent random variables $\{\xi_n,\ n = 1, 2, \ldots\}$ is called a GP if for some $u > 0$, the distribution of ξ_n is $F(u^{n-1}t)$. The constant u is called the parameter of the GP.

REMARK 2.1 From Definition 2, it follows that

(1) if $u > 1$, then $\{\xi_n,\ n = 1, 2, \ldots\}$ is stochastically decreasing: $\xi_n \geqslant \xi_{n+1}$,

(2) if $0 < u < 1$, then $\{\xi_n,\ n = 1, 2, \ldots\}$ is stochastically increasing: $\xi_n \leqslant \xi_{n+1}$ and

(3) if $u = 1$, then $\{\xi_n,\ n = 1, 2, \ldots\}$ is an RP.

We introduce a new process which can be used to describe scenarios with more complicated failure intensities.

DEFINITION 3 A sequence of non-negative independent random variables $\xi_n,\ n = 1, 2, \ldots$, is called an EPP if for some $\alpha + \beta \neq 0$, $\alpha, \beta \geqslant 0$, $a \geqslant 1$ and $0 < b \leqslant 1$, the cumulative distribution function (cdf) of ξ_n is $G((\alpha a^{n-1} + \beta b^{n-1})t)$, and $G(t)$ is an exponential cdf, where α, β, a and b are the parameters of the process.

REMARK 2.2 From Definition 3, it follows that

(1) if $a = b = 1$, then the EPP is an HPP,

(2) if $\alpha a^{n-1} \neq 0$ and $\beta b^{n-1} = 0$ (or $\alpha a = 0$ and $\beta b^{n-1} \neq 0$) for $n = 1, 2, \ldots$, then $\{\xi_n,\ n = 1, 2, \ldots\}$ is a GP,

(3) if $\alpha a^{n-1} \neq 0$ and $b = 1$, then $\{\xi_n,\ n = 1, 2, \ldots\}$ can model the failure pattern of the time from the random failure period to the wear-out failure period in the bathtub curve,

(4) if $a = 1$, $b < 1$ and $\beta b^{n-1} \neq 1$, then $\{\xi_n,\ n = 1, 2, \ldots\}$ can model the failure pattern of the time from the random failure period to the wear-out failure period in the bathtub curve and

(5) if $\alpha a^{n-1} \neq 0$, $a > 1$, $0 < b < 1$ and $\beta b^{n-1} \neq 0$, then $\{\xi_n,\ n = 1, 2, \ldots\}$ can model a system with a more complicated failure pattern.

Assume that the failure intensity of ξ_1 is λ, then the failure intensity of the variable ξ_n is

$$h_n = (\alpha a^{n-1} + \beta b^{n-1})\lambda. \tag{2.1}$$

For repairable systems, the Cox–Lewis model (Cox & Lewis, 1966) is a well-known NHPP model with the rate of occurrence of failures as follows:

$$\upsilon(t) = \mathrm{e}^{\alpha + \beta t}. \tag{2.2}$$

Apparently, if we discretise time t to cycle number n, the failure intensity in cycle n should be $h_n = \mathrm{e}^{\alpha+\beta n}\lambda$ (where $\alpha_0 = \mathrm{e}^{\alpha}$ and $a_0 = \mathrm{e}^{\beta}$). EPP models can therefore be regarded as a discretisation version of the Cox–Lewis model.

In applications, we can use at most three parameters of α, β, a and b because from the cases (1) to (4) in Remark 2.2, only three parameters are needed. In case (5) of Remark 2.2, one of the parameters, α or β, can be set to be 1 as the result is the same as the case when neither of the two parameters is 1.

Let $E(\xi_1) = \lambda^{-1}$, then

$$E(\xi_n) = ((\alpha a^{n-1} + \beta b^{n-1})\lambda)^{-1}. \tag{2.3}$$

3. Reliability indices

Denote $U_n = \sum_{i=1}^{n} \xi_i$ and $N(t) = \sup\{n\colon U_n < t\}$, and denote the cdf of U_n by $G_n(t)$. Therefore, $G_n(t)$ has a hypoexponential distribution shown as follows:

$$G_n(t) = \sum_{i=1}^{n} C_{i,n}(1 - \exp((\alpha a^{i-1} + \beta b^{i-1})\lambda t), \tag{3.1}$$

where

$$C_{i,n} = \prod_{j\neq i}^{n} \frac{\alpha a^{j-1} + \beta b^{j-1}}{\alpha(a^{j-1} - a^{i-1}) + \beta(b^{j-1} - b^{i-1})}.$$

Denote the expected number of failures and the expected system reliability within time interval (T_1, T_2) by $N(T_1, T_2)$ and $R(T_1, T_2)$, respectively.

PROPOSITION 3.1 The expected number of failures and the expected system reliability within time interval (T_1, T_2) are

$$N(T_1, T_2) = \sum_{n=1}^{\infty}\sum_{i=1}^{n} C_{i,n}(\exp(-(\alpha a^{i-1} + \beta b^{i-1})\lambda T_1) - \exp(-(\alpha a^{i-1} + \beta b^{i-1})\lambda T_2)) \tag{3.2}$$

and

$$R(T_1, T_2) = \exp(-N(T_1, T_2)), \tag{3.3}$$

respectively.

Proof. According to the theory of RP, we have

$$E(N(t)) = \sum_{n=1}^{\infty} G_n(t). \tag{3.4}$$

From Definition 3, the rate of occurrence of failures of the process is

$$\begin{aligned}\nu(t) &= \frac{\mathrm{d}E(N(t))}{\mathrm{d}t}\\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{n} C_{i,n}(\alpha a^{i-1} + \beta b^{i-1})\lambda \exp(-(\alpha a^{i-1} + \beta b^{i-1})\lambda t).\end{aligned} \tag{3.5}$$

Then, the expected number of failures within time interval (T_1, T_2) is

$$\begin{aligned} N(T_1, T_2) &= E\{N(T_2) - N(T_1)\} \\ &= \sum_{n=1}^{\infty} \{F(T_2) - F(T_1)\} \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{n} C_{i,n} (\exp(-(\alpha a^{i-1} + \beta b^{i-1})\lambda T_2) - \exp(-(\alpha a^{i-1} + \beta b^{i-1})\lambda T_1)). \end{aligned} \quad (3.6)$$

The expected system reliability in the interval (T_1, T_2) is

$$R(T_1, T_2) = \exp(-N(T_1, T_2)). \quad (3.7)$$

This proves Proposition 3.1. □

4. Optimal maintenance policies

This section considers two scenarios on maintenance policy: a policy where only corrective maintenance was conducted, and a policy where both corrective maintenance and preventive maintenance were performed. Some authors refer to CM as ‘repair’, and we will use them interchangeably in what follows.

Let X_n be the survival time after the $(n-1)$th repair, and Y_n be the repair time after the nth failure, where X_n and Y_n are independent. Assume that the cdfs of X_n and Y_n are $F((\alpha_1 a_1^{i-1} + \beta_1 b_1^{i-1})t)$ and $F((\alpha_2 a_2^{i-1} + \beta_2 b_2^{i-1})y)$, respectively. Denote costs for a replacement and corrective maintenance per time unit by C_{r} and C_{c}, respectively. Denote the business profit per time unit by C_{b}.

4.1 *Maintenance policy 1*

Consider a maintenance policy: a system is replaced with a new one if the expected cost reaches the minimal value in its lifetime. Assume that repair is carried out as soon as the system fails, and the system is started as soon as the repair is completed. Suppose the replacement time is neglectable.

Assume that there are $N-1$ repairs before the system is replaced. Then, the expected time span is

$$T_N = \sum_{n=1}^{N} \frac{1}{(\alpha_1 a_1^{i-1} + \beta_1 b_1^{i-1})\lambda} + \sum_{n=1}^{N-1} \frac{1}{(\alpha_2 a_2^{i-1} + \beta_2 b_2^{i-1})\mu}. \quad (4.1)$$

The long-run average cost per time unit

$$\frac{1}{T_N}\left(\sum_{n=1}^{N-1} \frac{C_{\mathrm{r}}}{(\alpha_2 a_2^{i-1} + \beta_2 b_2^{i-1})\mu} - \sum_{n=1}^{N} \frac{C_{\mathrm{b}}}{(\alpha_1 a_1^{i-1} + \beta_1 b_1^{i-1})\lambda} + C_{\mathrm{p}}\right). \quad (4.2)$$

4.2 *Maintenance policy 2*

Consider another maintenance policy which has also been considered by Zhang (2002). Zhang (2002) optimises the long-run average cost when preventive maintenance is considered. The failure processes after both corrective maintenance and preventive maintenance are assumed to be GP. He assumes that a preventive maintenance activity is performed as soon as the operating time of a system reaches a pre-specified time τ, or a corrective maintenance is conducted upon failure within time interval τ, whichever

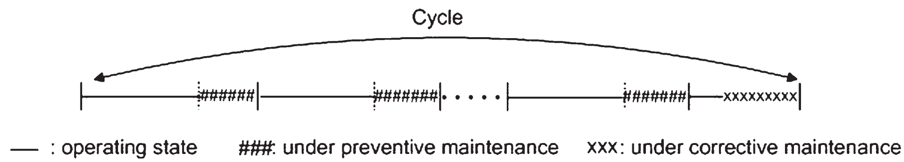

FIG. 2. Maintenance policy 2.

occurs first. Zhang (2002) defines a cycle to be the time from the end of a corrective maintenance activity to the end of the next corrective maintenance activity . A preventive maintenance activity is assumed to restore the system to be 'as good as' the state just when the system is started in the same cycle. A possible scenario is shown in Fig. 2.

Denote the survival time after the $(n-1)$th preventive maintenance by X'_n. X'_n should be longer than the survival time of the system without any preventive maintenance, X_n. Denote the cdf of X'_n by $H_n(t)$. The relationship between X'_n and X_n is shown as follows:

If $k\tau \leqslant t < (k+1)\tau$, we have

$$\begin{aligned} H_n(t) &= \Pr\{X'_n < t\} \\ &= \Pr\{k\tau + X_n \chi\{X_n < \tau\} < t\} \\ &= F_n(t-k\tau)(1-F_n(\tau))^k, \end{aligned} \tag{4.3}$$

where $\chi\{\cdot\}$ is an indicator function satisfying $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. The expected value of X'_n can be obtained as follows:

$$\begin{aligned} E(X'_n) &= \int_0^\infty t\,\mathrm{d}H_n(t) \\ &= \sum_{k=0}^{\infty} \int_{k\tau}^{(k+1)\tau} t(1-F_n(\tau))^k\,\mathrm{d}F_n(t-k\tau) \\ &= \sum_{k=0}^{\infty} \int_0^{\tau} (y+k\tau)(1-F_n(\tau))^k\,\mathrm{d}F_n(y) \\ &= \frac{1}{F_n(\tau)}\left(\int_0^\tau y\,\mathrm{d}F_n(y) + \tau(1-F_n(\tau))\right). \end{aligned} \tag{4.4}$$

Zhang (2002) derives the following result:

$$E(X'_n) = \int_0^\tau y\,\mathrm{d}F_n(y) + \frac{\tau(1-F_n(\tau))}{F_n(\tau)}. \tag{4.5}$$

Compared with (4.4), the result in (4.5) from Zhang (2002) is not correct. Hence, the cost expression $D(N)$ in Zhang (2002) should be changed.

Assume that the expected time on a preventive maintenance activity is μ_{p}. Denote cost on a preventive maintenance activity per time unit by C_{p}, and the number of preventive maintenance activities in cycle n is N_n. Therefore, the expected time on preventive maintenance activities is $\mu_{\mathrm{p}} N_n$.

TABLE 1 *Engine failure data*

Repair times	Mean time between failures (miles)
1st	94000
2nd	70000
3rd	54000
4th	41000
5th	33000

The expected time span of N' cycles is

$$T'_N = \sum_{n=1}^{N'} \left(\int_0^{\tau} y \,\mathrm{d}F_n(y) + \frac{\tau(1 - F_n(\tau))}{F_n(\tau)} + \frac{1}{(\alpha_2 a_2^{n-1} + \beta_2 b_2^{n-1})\mu} + \mu_{\mathrm{p}} N_n \right). \tag{4.6}$$

Then we need to minimise the following measure to obtain the optimal cycle N' or the optimal time interval τ.

$$\frac{1}{T'_N} \sum_{n=1}^{N'} \left[-\left(\int_0^{\tau} y \,\mathrm{d}F_n(y) + \frac{\tau(1 - F_n(\tau))}{F_n(\tau)} \right) C_{\mathrm{b}} + \frac{C_{\mathrm{c}}}{(\alpha_2 a_2^{n-1} + \beta_2 b_2^{n-1})\mu} + \mu_{\mathrm{p}} N_n C_{\mathrm{p}} + C_{\mathrm{r}} \right]. \tag{4.7}$$

5. Estimating parameters and a case study

For estimating the parameters in an EPP, both the maximum likelihood algorithm and the least-square algorithm can be used.

For example, for an EPP $\{X_n,\ n = 1, 2, \ldots\}$, we make use of the least-squared algorithm to estimate the parameters. Assume $\{x_n,\ n = 1, 2, \ldots M\}$ is a sequence of data which is the time to failure in each cycle. We can minimise the following least-squared function,

$$\sum_{n=1}^{M} (x_n - [(\alpha^{n-1} + \beta b^{n-1})\lambda]^{-1})^2, \tag{5.1}$$

to obtain the estimates of the parameters.

Consider the bus engine failure data set in Table 1 given by Davis (1952). Assume that the repairs are normal so that the system is not as good as new after each repair. If both the EPP model and the GP model are used to fit the data, the following results are obtained.

Taking the EPP with parameters a, b and let $\alpha = 1$, $\beta = 1$, the least-squares estimates of a, b and the failure intensity λ are $\hat{a} = 0.71299$, $\hat{b} = 0.00611$ and $\hat{\lambda} = 0.00001214$, respectively The residual mean-squared error (MSE) is 155.43. Taking the GP with parameter u, the least-squares estimates of $\hat{\lambda} = 0.000010729$, $\hat{u} = 1.3101$, with a much larger residual MSE of 404.32.

It shows that an EPP model is more suitable in this example than a GP model.

6. Concluding remarks

The application of GP is rather restricted in the sense that it can only describe a system with either an increasing or a decreasing failure intensity. The new process introduced in this paper can model repairable systems with more complicated failure intensities. The long-run average cost under the two maintenance policies, a policy where only corrective maintenance is performed and a policy where both corrective maintenance and preventive maintenance are executed, are derived. We also updated Zhang's result (Zhang, 2002) in this paper.

Baker & Christer (1994) pointed out that most models for failure intensity in the reliability literature were 'the lack of evident conviction in applicability to real-world situations manifest by no indication of how the values of model parameters can be determined and no examples of actual applications or case studies or post-modelling analysis'. The model introduced in this paper addresses these drawbacks by providing parameter estimation and a case study. Our further work will compare this model with other models such as HPP and NHPP.

Acknowledgements

The authors would like to thank EPSRC as part of the Innovative Manufacturing Research Centres initiative for their financial support and our industrial partners (EC Harris, Dytecna, EMCOR Rail, INBIS and Quorum Logistics Support). Thanks are also due to Dr Wenbin Wang from University of Salford and the referees for their helpful comments and suggestions, which have resulted in a number of improvements in the paper.

REFERENCES

ASCHER, H. E. & FEINGOLD, H. (1984a) *Repairable System Modelling, Inference, Misconceptions and Their Causes*. New York: Marcel Dekker.

BAKER, R. D. & CHRISTER, A. H. (1994) Review of delay-time or modelling of engineering aspects of maintenance. *Eur. J. Oper. Res.*, **73**, 407–422.

BLOCK, H. W., BORGES, W. S. & SAVITS, T. H. (1985) Age-dependent minimal repair. *J. Appl. Probab.*, **20**, 851–859.

BROWN, M. & PROSCHAN, F. (1983) Imperfect repair. *J. Appl. Probab.*, **20**, 851–859.

COX, D. R. & LEWIS, P. A. (1966) *The Statistical Analysis of Series of Events*. London: Methuen.

DAVIS, D. J. (1952) An analysis of some failure data. *J. Am. Stat. Soc.*, **47**, 113–150.

LAM, Y. (1988) Geometric processes and replacement problem. *Acta Math. Appl. Sin.*, **20**, 479–482.

LAM, Y. & ZHANG, Y. L. (2003) A geometric-process maintenance model for a deteriorating system under a random environment. *IEEE Trans. Reliab.*, **52**, 83–88.

LAM, Y., ZHANG, Y. L. & ZHENG, Y. H. (2002) A geometric process equivalent model for a multistate degenerative system. *Eur. J. Oper. Res.*, **142**, 21–29.

LINDQVIST, B. H., ELVEBAKK, G. & HEGGLAND, K. (2003) The trend-renewal process for statistical analysis of repairable systems. *Technometrics*, **45**, 31–44.

PHAM, H. & WANG, H. (1996) Imperfect maintenance. *Eur. J. Oper. Res.*, **94**, 425–438.

ROSS, S. M. (1996) *Stochastic Processes*. New York: Wiley.

SCARF, P. A. (1997) On the application of mathematical models in maintenance. *Eur. J. Oper. Res.*, **99**, 493–506.

WANG, H. (2002) A survey of maintenance policies of deteriorating systems. *Eur. J. Oper. Res.*, **139**, 469–489.

WANG, H. & PHAM, H. (1996) A quasi renewal process and its applications in imperfect maintenance. *Int. J. Syst. Sci.*, **27**, 1055–1062.

WU, S. M., HUANG, R. & WAN, D. J. (1994) Reliability analysis of a repairable system without being repaired "as good as new". *Microelectron. Reliab.*, **34**, 357–360.

YANEZ, M., JOGLAR, F. & MODARRES, M. (2002) Generalized renewal process for analysis of repairable systems with limited failure experience. *Reliab. Eng. Syst. Saf.*, **77**, 167–180.

ZHANG, Y. L. (1999) An optimal geometric process model for a cold standby repairable system. *Reliab. Eng. Syst. Saf.*, **63**, 107–110.

ZHANG, Y. L. (2002) A geometric-process repair-model with good-as-new preventive repair. *IEEE Trans. Reliab.*, **51**, 223–228.