Computational Statistics and Data Analysis 53 (2008) 450–462

Contents lists available at ScienceDirect

Computational Statistics and Data Analysis

journal homepage: www.elsevier.com/locate/csda

A generalized modified Weibull distribution for lifetime modeling

Jalmar M.F. Carrasco [a], Edwin M.M. Ortega [a,*], Gauss M. Cordeiro [b]

[a] ESALQ, Universidade de São Paulo, Piracicaba, Brazil
[b] DEINFO, Universidade Federal Rural de Pernambuco, Brazil

ARTICLE INFO

Article history:
Received 13 May 2008
Received in revised form 11 August 2008
Accepted 14 August 2008
Available online 24 August 2008

ABSTRACT

A four parameter generalization of the Weibull distribution capable of modeling a bathtub-shaped hazard rate function is defined and studied. The beauty and importance of this distribution lies in its ability to model monotone as well as non-monotone failure rates, which are quite common in lifetime problems and reliability. The new distribution has a number of well-known lifetime special sub-models, such as the Weibull, extreme value, exponentiated Weibull, generalized Rayleigh and modified Weibull distributions, among others. We derive two infinite sum representations for its moments. The density of the order statistics is obtained. The method of maximum likelihood is used for estimating the model parameters. Also, the observed information matrix is obtained. Two applications are presented to illustrate the proposed distribution.

1. Introduction

The Weibull distribution, having exponential and Rayleigh as special cases, is a very popular distribution for modeling lifetime data and for modeling phenomenon with monotone failure rates. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, the Weibull distribution does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates such as the bathtub-shaped and the unimodal failure rates which are common in reliability and biological studies. Such bathtub hazard curves have nearly flat middle portions and the corresponding densities have a positive anti-mode. An example of bathtub-shaped failure rate is the human mortality experience with a high infant mortality rate which reduces rapidly to reach a low. It then remains at that level for quite a few years before picking up again. Unimodal failure rates can be observed in course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

The models that present bathtub-shaped failure rate are very useful in survival analysis. But, according to Nelson (1982), the distributions presented in lifetime literature with this type of data, as the distributions proposed by Hjorth (1980), are sufficiently complex and, therefore, difficult to be modeled. Later, other works had introduced new distributions for modeling bathtub-shaped failure rate. For example, Rajarshi and Rajarshi (1988) presented a review of these distributions and Haupt and Schabe (1992) considered a lifetime model with bathtub failure rates. However, these models do not present much practicability to be used and in recent years new classes of distributions were proposed based on modifications of the Weibull distribution to cope with bathtub-shaped failure rate. Among these, the exponentiated Weibull (EW) distribution introduced by Mudholkar et al. (1995, 1996), the additive Weibull distribution (Xie and Lai, 1995), the extended Weibull distribution presented by Xie et al. (2002), the modified Weibull (MW) distribution proposed by Lai et al. (2003) and the extended flexible Weibull distribution by Bebbington et al. (2007), among others. A good review of these models is presented in Pham and Lai (2007).

* Corresponding address: Departamento de Ciências Exatas, ESALQ/USP, Av. Pádua Dias 11 - Caixa Postal 9, 13418-900, Piracicaba - São Paulo, Brazil. Tel.: +55 11 19 34294127; fax: +55 11 19 34294468.
E-mail addresses: jalmar@esalq.usp.br (J.M.F. Carrasco), edwin@esalq.usp.br (E.M.M. Ortega), gausscordeiro@uol.com.br (G.M. Cordeiro).

0167-9473/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.csda.2008.08.023

In this work we present a new distribution called the generalized modified Weibull (GMW) distribution with four parameters. The new distribution due to its flexibility in accommodating all the forms of the hazard rate function can be used in a variety of problems for modeling lifetime data. Another important characteristic of the distribution is that it contains, as special sub-models, the Weibull, exponentiated exponential (Gupta and Kundu, 1999, 2001), EW (Mudholkar et al., 1995, 1996), generalized Rayleigh (Kundu and Rakab, 2005), MW (Lai et al., 2003) and some other distributions. The GMW distribution is not only convenient for modeling comfortable bathtub-shaped failure rates data but is also suitable for testing goodness-of-fit of some special sub-models such as the EW and MW distributions.

The rest of the paper is organized as follows. In Section 2, we introduce the GMW distribution. Some special cases are presented in Section 3. Section 4 gives two general formulae for the moments. In Section 5 we provide an expression for the density function of the order statistics and a simple way to compute their moments. The total time on test (TTT) transform procedure is used in Section 6 as a tool to identify the hazard behavior of the distribution. Estimation by maximum likelihood and testing of hypotheses are outlined in Section 7. Two lifetime data sets are used in Section 8 to illustrate the usefulness of the distribution. Section 9 ends with some conclusions.

2. A generalized modified Weibull distribution

Let $F_L(t;\alpha,\gamma,\lambda)$ be the cumulative distribution function (cdf) of the MW distribution proposed by Lai et al. (2003). The cdf of the GMW distribution can be defined by elevating $F_L(t;\alpha,\gamma,\lambda)$ to the power of β, namely $F(t) = F_L(t;\alpha,\gamma,\lambda)^{\beta}$. Hence, the density function of the GMW distribution with four parameters $\alpha > 0$, $\gamma \geq 0$, $\lambda \geq 0$ and $\beta > 0$ is given by

$$f(t) = \frac{\alpha\beta t^{\gamma-1}(\gamma+\lambda t)\exp\{\lambda t - \alpha t^{\gamma}\exp(\lambda t)\}}{[1-\exp\{-\alpha t^{\gamma}\exp(\lambda t)\}]^{1-\beta}}, \quad t > 0. \tag{1}$$

We can easily prove that (1) is a density function by considering the substitution $u = 1 - \exp\{-\alpha t^{\gamma}\exp(\lambda t)\}$. The parameter α controls the scale of the distribution, whereas the parameters γ and β control its shape. The parameter λ is a kind of accelerating factor in the imperfection time and it works as a factor of fragility in the survival of the individual when the time increases. The Weibull distribution is a special case of (1) when $\lambda = 0$ and $\beta = 1$. If X is a random variable with density (1), we write $X \sim \text{GMW}(\alpha,\gamma,\lambda,\beta)$.

The corresponding survival and the hazard rate functions are, respectively,

$$S(t) = 1 - F(t) = 1 - [1-\exp\{-\alpha t^{\gamma}\exp(\lambda t)\}]^{\beta} \tag{2}$$

and

$$h(t) = \frac{\alpha\beta t^{\gamma-1}(\gamma+\lambda t)\exp\{\lambda t - \alpha t^{\gamma}\exp(\lambda t)\}[1-\exp\{-\alpha t^{\gamma}\exp(\lambda t)\}]^{\beta-1}}{1-[1-\exp\{-\alpha t^{\gamma}\exp(\lambda t)\}]^{\beta}}. \tag{3}$$

The new distribution has a closed-form survival function and is quite flexible for modeling survival data. We simulate the GMW distribution by solving the nonlinear equation

$$t^{\gamma}\exp(\lambda t) + \frac{1}{\alpha}\log(1-u^{1/\beta}) = 0, \tag{4}$$

where u has the uniform $U(0,1)$ distribution.

Fig. 1 plots some GMW density curves for different choices of the parameters γ and β. The Weibull distribution corresponds to the particular choice $\lambda = 0$ and $\beta = 1$.

3. Special distributions

The GMW distribution contains as special sub-models the following well-known distributions:

- **Weibull distribution** For $\lambda = 0$ and $\beta = 1$, Eq. (1) gives

  $$f(t) = \alpha\gamma t^{\gamma-1}\exp(-\alpha t^{\gamma}), \quad t > 0,$$

  which is the classical two parameter Weibull distribution. If $\gamma = 1$ and $\gamma = 2$ in addition to $\lambda = 0$, $\beta = 1$, it coincides with the exponential and Rayleigh distributions, respectively.
- **Extreme-value distribution** For $\gamma = 0$ and $\beta = 1$, (1) gives

  $$f(t) = \alpha\lambda\exp\{\lambda t - \alpha\exp(\lambda t)\}, \quad t > 0,$$

  which is a type I extreme-value (also known as a log-gamma) distribution.
- **Exponentiated Weibull distribution** For $\lambda = 0$, the GMW distribution reduces to

  $$f(t) = \alpha\beta\gamma t^{\gamma-1}\exp(-\alpha t^{\gamma})\{1-\exp(-\alpha t^{\gamma})\}^{\beta-1}, \quad t > 0,$$

  which is the density of the EW distribution introduced by Mudholkar et al. (1995, 1996). If $\gamma = 1$ in addition to $\lambda = 0$, the GMW distribution becomes the exponentiated exponential distribution (see, Gupta and Kundu (1999, 2001)). If $\gamma = 2$ in addition to $\lambda = 0$, the GMW distribution becomes the generalized Rayleigh distribution (Kundu and Rakab, 2005).

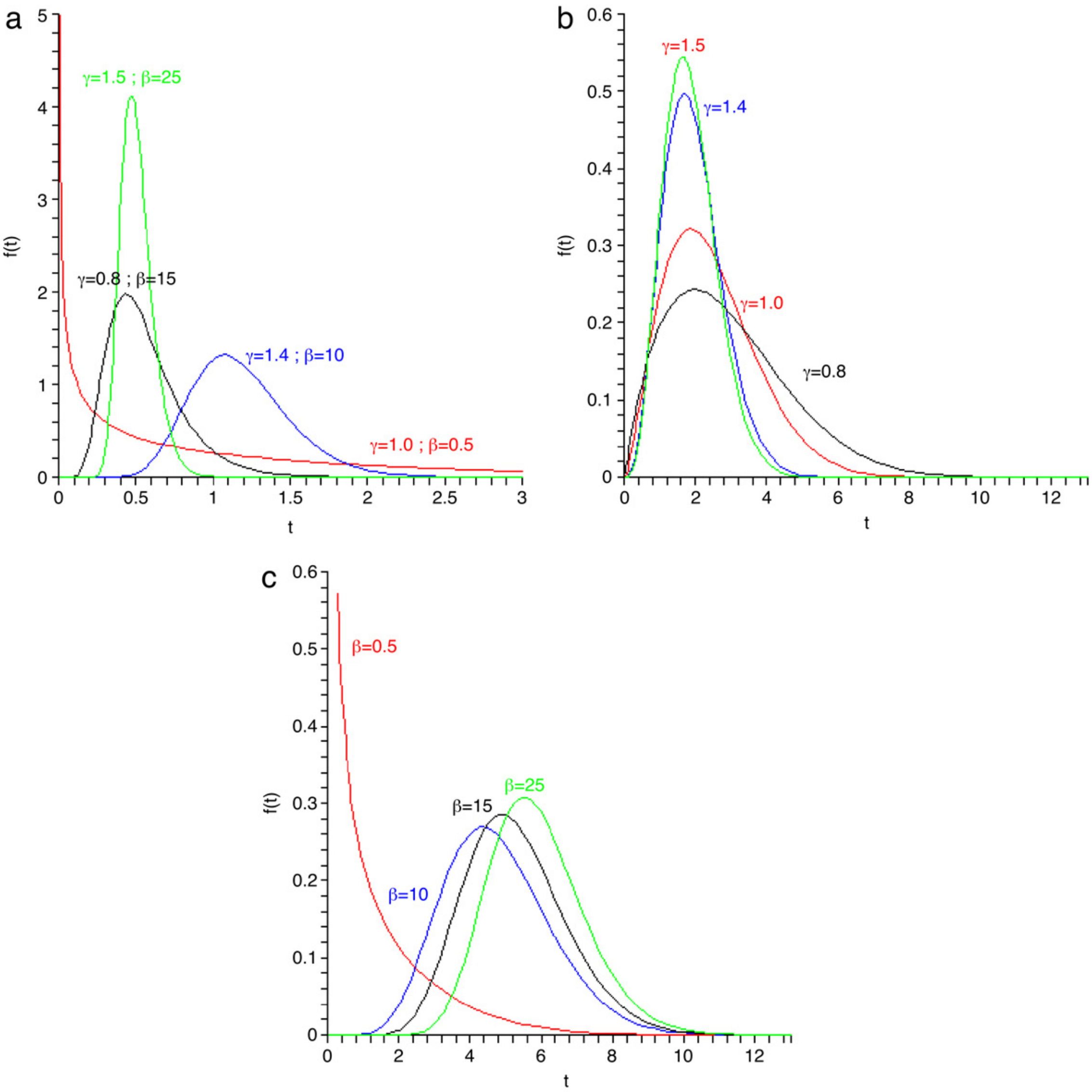

Fig. 1. GMW density curves. (a) Both parameters γ and β vary. (b) Only the parameter γ varies. (c) Only the parameter β varies.

- **Modified Weibull distribution** For $\beta = 1$, the GMW distribution reduces to

$$f(t) = \alpha t^{\gamma-1}(\gamma + \lambda t)\exp\{\lambda t - \alpha t^{\gamma}\exp(\lambda t)\}, \quad t > 0,$$

which is the density of the MW distribution introduced by Lai et al. (2003).

- **Beta integrated distribution** The beta integrated distribution was defined by Lai et al. (1998) and its survival function is

$$S(t) = \exp\{-\alpha t^{\gamma}(1 - \mathrm{d}t)^{c}\}, \quad \alpha, \gamma, d > 0, c < 0.$$

Fixing $d = 1/n$ and $c = -\lambda n$ and taking $n \to \infty$, yields

$$(1 - \mathrm{d}t)^{c} = \left(1 - \frac{t}{n}\right)^{-\lambda n} \to \exp(\lambda t)$$

and in the limit

$$S(t) \approx \exp\{-\alpha t^{\gamma}\exp(\lambda t)\},$$

which is the survival function (2) of the GMW distribution when $\beta = 1$.

4. General formulae for the moments

We hardly need to emphasize the necessity and importance of moments in any statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis). We now derive an infinite sum representation for the rth ordinary moment $\mu'_r = E(T^r)$ of the GMW distribution. We have

$$\mu'_r = \alpha\beta \int_0^\infty t^{r+\gamma-1}(\gamma+\lambda t)\exp\{\lambda t - \alpha t^\gamma \exp(\lambda t)\}\left[1-\exp\{-\alpha t^\gamma \exp(\lambda t)\}\right]^{\beta-1}\,\mathrm{d}t.$$

We use the series representation (assuming β real non-integer)

$$(1+z)^{\beta-1} = \sum_{j=0}^{\infty} \frac{\Gamma(\beta)z^j}{\Gamma(\beta-j)j!}.$$

If β is an integer, we can work with the binomial expansion. The last term in μ'_r can be expanded (for β real non-integer) to yield

$$\mu'_r = \alpha\beta\Gamma(\beta)\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(\beta-j)j!}\int_0^\infty t^{r+\gamma-1}(\gamma+\lambda t)\mathrm{e}^{\lambda t}\exp\{-(j+1)\alpha t^\gamma \mathrm{e}^{\lambda t}\}\mathrm{d}t. \quad (5)$$

Letting $x = t^\gamma \mathrm{e}^{\lambda t}$ we can invert to obtain t in terms of x when both λ and γ are positive by

$$t = \frac{\gamma}{\lambda}F\left(\frac{\lambda x^{1/\gamma}}{\gamma}\right), \quad (6)$$

where

$$F(z) = \sum_{i=1}^{\infty}\frac{(-1)^{i+1}i^{i-2}z^i}{(i-1)!}.$$

We have checked (6) and the above power series expansion for $F(z) = ProductLog[z]$ using the software Mathematica that gives $F(z)$ as the principal solution for w in $z = w\,\mathrm{e}^w$. We obtain

$$F(z) = z - z^2 + \frac{3z^3}{2} - \frac{8z^4}{3} + \frac{125z^5}{24} - \frac{54z^6}{5} + \frac{16\,807z^7}{720} - \frac{16\,384z^8}{315} + \frac{531\,441z^9}{4480} - \frac{156\,250z^{10}}{567}$$
$$+ \frac{2\,357\,947\,691z^{11}}{3\,628\,800} - \frac{2\,985\,984z^{12}}{1925} + \frac{1\,792\,160\,394\,037z^{13}}{479\,001\,600} - \frac{7\,909\,306\,972z^{14}}{868\,725} + O\left(z^{15}\right).$$

Hence, we can express t in terms of x from (6) by

$$t = \sum_{i=1}^{\infty} a_i x^{i/\gamma},$$

where

$$a_i = \frac{(-1)^{i+1}i^{i-2}}{(i-1)!}(\lambda/\gamma)^{i-1}. \quad (7)$$

Changing the variable t by x, the last integral in (5) becomes

$$I = \int_0^\infty \left\{\sum_{i=1}^{\infty} a_i x^{i/\gamma}\right\}^r \exp\{-(j+1)\alpha x\}\mathrm{d}x.$$

But

$$\left\{\sum_{i=1}^{\infty} a_i x^{i/\gamma}\right\}^r = \sum_{i_1,\ldots,i_r=1}^{\infty} A_{i_1,\ldots,i_r} x^{s_r/\gamma},$$

where

$$A_{i_1,\ldots,i_r} = a_{i_1}\ldots a_{i_r} \quad \text{and} \quad s_r = i_1 + \cdots + i_r.$$

Then, I can be rewritten as

$$I = \sum_{i_1,\ldots,i_r=1}^{\infty} A_{i_1,\ldots,i_r}\int_0^\infty x^{s_r/\gamma}\exp\{-(j+1)\alpha x\}\mathrm{d}x.$$

Substituting $y = (j+1)\alpha x$, we can obtain

$$I = \sum_{i_1,\ldots,i_r=1}^{\infty} \frac{A_{i_1,\ldots,i_r}\Gamma(s_r/\gamma+1)}{\{\alpha(j+1)\}^{s_r/\gamma+1}}.$$

Returning to (5) yields

$$\mu_r' = \alpha\beta \sum_{j=0}^{\infty} \frac{(1-\beta)_j}{j!} \sum_{i_1,\ldots,i_r=1}^{\infty} \frac{A_{i_1,\ldots,i_r}\Gamma(s_r/\gamma+1)}{\{\alpha(j+1)\}^{s_r/\gamma+1}}, \tag{8}$$

where $(1-\beta)_j = (-1)^j\Gamma(\beta)/\Gamma(\beta-j)$ and the term $A_{i_1,\ldots,i_r}$ can be easily computed from (7). Formula (8) for the rth moment of the GMW distribution is quite general and holds when both parameters λ and γ are positive and $\beta \neq 1$. An alternative expression for μ_r' comes from (7) and (8) as

$$\mu_r' = \alpha\beta \sum_{j=0}^{\infty} \frac{(1-\beta)_j}{j!\{\alpha(j+1)\}^{r/\gamma+1}} \sum_{i_1,\ldots,i_r=0}^{\infty} \frac{\Gamma((i_1+\cdots+i_r+r)/\gamma+1)}{(i_1+1)^{1-i_1}\cdots(i_r+1)^{1-i_r}} \frac{x_1^{i_1}\cdots x_r^{i_r}}{i_1!\cdots i_r!}, \tag{9}$$

where $x_i = (-\lambda/\gamma)\{\alpha(j+1)\}^{-1/\gamma}$.

Eqs. (8) and (9) represent the main results of this section. The advantage of these expressions is that they can be used to determine moments without any restrictions or conditions on the four parameters. The case $\lambda = 0$ is already available in the literature (see the examples below) and the case $\beta = 1$ is also discussed below. The complexity of formulae (8) and (9) increase when r increases. The first moment from (8) becomes

$$\mu_1' = \alpha\beta \sum_{j=0}^{\infty} \frac{(1-\beta)_j}{j!} \sum_{i=1}^{\infty} \frac{a_i\Gamma(i/\gamma+1)}{\{\alpha(j+1)\}^{i/\gamma+1}}.$$

Further moments, skewness and kurtosis can be obtained from (8) or (9).

When $\lambda = 0$, Choudhury (2005), Nadarajah (2005) and Nadarajah and Kotz (2005) show that the moments of the EW distribution for any $r > -\gamma$ (both real and integer) are (which follow easily from (8))

$$\mu_r' = E(T^r) = \beta\alpha^{-r/\gamma}\Gamma(r/\gamma+1) \sum_{i=0}^{\infty} \frac{(1-\beta)_i}{i!(i+1)^{(r+\gamma)/\gamma}}.$$

Various closed-form expressions can be obtained as special cases of this formula. In fact, if $\gamma = 1$ in addition to $\lambda = 0$, the moments agree with those of the exponentiated exponential distribution that come from the moment generating function $M_T(t)$ of T (Gupta and Kundu, 1999, 2001)

$$M_T(t) = E(e^{Tt}) = \frac{\Gamma(\beta+1)\Gamma(1-t/\alpha)}{\Gamma(\beta-t/\alpha+1)}, \quad t < \alpha. \tag{10}$$

Therefore, it immediately follows that

$$E(T) = \alpha^{-1}\{\psi(\beta+1)-\psi(1)\} \quad \text{and} \quad \mathrm{Var}(T) = \alpha^{-2}\{\psi'(1)-\psi'(\beta+1)\},$$

where $\Gamma(\cdot)$ is the gamma function and $\psi(\cdot)$ and its derivatives are the digamma and polygamma functions. The mean of the exponentiated exponential distribution is increasing to ∞ as the shape parameter β increases, for fixed α. Also, for fixed α, the variance increases with β and it increases to $\pi^2/(6\alpha)$. This feature is quite different compared to gamma and Weibull distributions. In case of the gamma distribution, the variance goes to ∞ when β increases, whereas for the Weibull distribution the variance is approximately $\pi^2/(6\alpha\beta^2)$ for large values of β.

Further, if $\gamma = 2$ in addition to $\lambda = 0$, Kundu and Rakab (2005) observed that the moments of the generalized Rayleigh distribution cannot be expressed in a nice form but they showed that

$$E(T^2) = \alpha^{-2}\{\psi(\beta+1)-\psi(1)\} \quad \text{and} \quad \mathrm{Var}(T^2) = \alpha^{-4}\{\psi'(1)-\psi'(\beta+1)\}.$$

Finally, the rth moment for $\beta = 1$ follows the same lines of the general proof. We can obtain

$$\mu_r' = \sum_{i_1,\ldots,i_r=1}^{\infty} \frac{A_{i_1,\ldots,i_r}\Gamma(s_r/\gamma+1)}{\alpha^{s_r/\gamma}},$$

which is a new result for the moments of the MW distribution.

5. Order statistics

We now give the density of the ith order statistic $X_{i:n}, f_{i:n}(x)$ say, in a random sample of size n from the GMW distribution. It is well known that (for $i = 1, \ldots, n$)

$$f_{i:n}(t) = \frac{1}{B(i, n-i+1)} f(t) F^{i-1}(t) \{1 - F(t)\}^{n-i},$$

where $B(i, n-i+1)$ is the beta function.

From (1) and (2) we can write

$$f_{i:n}(t) = \frac{\alpha\beta t^{\gamma-1}(\gamma+\lambda t)\exp\{\lambda t - \alpha t^{\gamma}\exp(\lambda t)\}\left[1-\{1-\exp(-\alpha t^{\gamma}\exp(\lambda t))\}^{\beta}\right]^{n-i}}{B(i, n-i+1)\left[1-\exp\{-\alpha t^{\gamma}\exp(\lambda t)\}\right]^{1-\beta i}}.$$

Using the binomial expansion we can express the density function of the ith order statistic as a finite weighted sum of densities of the GMW distributions

$$f_{i:n}(t) = \sum_{k=0}^{n-i} w_{i,k} f_{i+k}(t), \tag{11}$$

where $f_{i+k}(t)$ is the density function of the GMW$(\alpha, \gamma, \lambda, \beta(i+k))$ distribution and the weights are simply $w_{i,k} = (-1)^k \binom{i+k-1}{k}\binom{n}{i+k}$. From (11) we can obtain the rth moment of the ith order statistic using the general formulae (8) and (9) for the rth moment of the GMW distribution.

6. Hazard rate function shapes

Glaser (1980) gave sufficient conditions to characterize a given failure rate distribution as being bathtub shaped, increasing and decreasing and upside-down bathtub. A characteristic of the GMW distribution is that its failure rate function accommodates all these shapes that depend basically on the values of the parameters γ and β. The study of the hazard rate function shape involves an analysis of the first derivative $h'(t) = \mathrm{d}h(t)/\mathrm{d}t$ given by

$$h'(t) = h(t)\left[\frac{\gamma-1}{t} + \lambda\left(\frac{1}{\gamma+t\lambda}+1\right) - \frac{\log(u)}{(1-u)}\left(\frac{\gamma}{t}+\lambda\right)\left\{1 - \frac{\beta u}{1-(1-u)^{\beta}}\right\}\right], \tag{12}$$

where $u = \exp\{-\alpha t^{\gamma}\exp(\lambda t)\}$. Note that $0 < u < 1$, then $-\log(u) > 0$, $0 < 1-u < 1$ and $0 < 1-(1-u)^{\beta} < 1$. From (12), it can be verified that

- For $\gamma \geq 1$, $0 < \beta < 1$ and $\forall t > 0$, $h'(t) > 0$, $h(t)$ is increasing;
- For $0 < \gamma < 1$, $\beta > 1$ and $\forall t > 0$, $h'(t) < 0$, $h(t)$ is decreasing;
- For $0 < \gamma < 1$ and $\beta \to \infty$, $h(t)$ is unimodal;
- If $\lambda = 0$, $\gamma > 1$ and $\gamma\beta < 1$, $h(t)$ is bathtub shaped;
- If $\beta = 1$, we have $h'(t) = \alpha t^{\gamma-1}\exp(\lambda t)[(\gamma+\lambda t)\{(\gamma-1)t^{-1}+\lambda\}+\lambda] = 0$, and by solving this equation, a change point can be obtained as $t^* = (-\gamma+\sqrt{\gamma})/\lambda$. It can be seen that when $0 < \gamma < 1$, t^* exists and is finite. When $t < t^*$, $h'(t^*) < 0$, the hazard rate function is decreasing; when $t > t^*$, $h'(t^*) > 0$, the hazard rate function is increasing. Hence, the hazard rate function can be of bathtub shape.

The distinct types of hazard shapes of the GMW distribution are illustrated in Fig. 2 for some selected parameter values.

In order to identify the type of failure rate of the lifetime data, many approaches have been proposed (see, Glaser (1980)). A graphical method based on the TTT transform introduced by Barlow and Campo (1975) will be used as a tool to illustrate the variety of the hazard rate shapes. For this purpose, the empirical TTT transform $H_F^{-1}(r/n)$ can be used as a tool to identify the hazard shape for a given data set. The scaled empirical TTT is given by

$$\phi_n\left(\frac{r}{n}\right) = \frac{H_F^{-1}(r/n)}{H_F^{-1}(1)} = \frac{\sum_{i=1}^{r} T_{i:n} + (n-r)T_{r:n}}{\sum_{i=1}^{n} T_i}, \tag{13}$$

where $r = 1, \ldots, n$ and $T_{i:n}$, for $i = 1, \ldots, n$, represent the order statistics of the sample. If the empirical TTT transform is convex, concave, convex then concave and concave then convex, the shape of the corresponding hazard rate function for such failure data is, respectively, decreasing, increasing, bathtub and unimodal (Barlow and Campo, 1975; Aarset, 1987; Mudholkar et al., 1996). Then, the TTT-transform can illustrate the variety of the hazard rate curves for a lifetime distribution.

The scaled TTT transform for the GMW distribution can be defined as

$$\phi_F(u) = \frac{H_F^{-1}(u)}{H_F^{-1}(1)}, \quad 0 < u < 1, \tag{14}$$

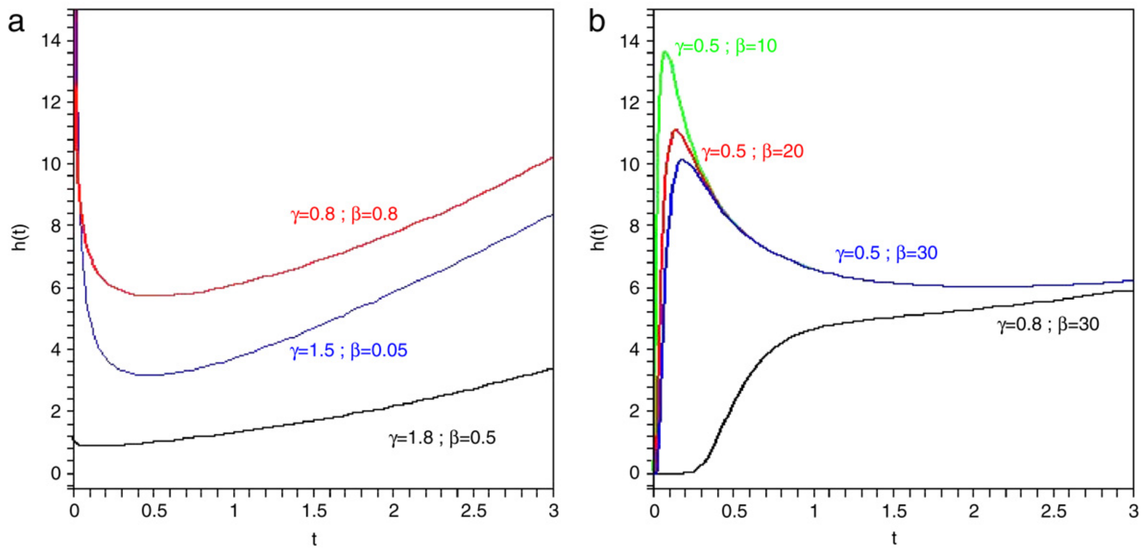

Fig. 2. GMW hazard rate functions. (a) The distribution has a bathtub hazard rate function. (b) The distribution has an unimodal failure rate.

where $H_F^{-1}(u) = \int_0^{F^{-1}(u)}\{1 - F(x)\}dx$ and the function $F^{-1}(u)$ is obtained numerically from (4). In order to obtain $\phi_F(u)$ and the scaled TTT-transform, the integrals are calculated numerically from the survival function (2). Plots of the scaled TTT-transforms for the GMW distribution to show its model flexibility are given in Fig. 3.

7. Maximum likelihood estimation

Let T_i be a random variable following (1) with the vector of parameters $\boldsymbol{\theta} = (\alpha, \gamma, \lambda, \beta)^T$. The data encountered in survival analysis and reliability studies are often censored. A very simple random censoring mechanism that is often realistic is one in which each individual i is assumed to have a lifetime T_i and a censoring time C_i, where T_i and C_i are independent random variables. Suppose that the data consist of n independent observations $t_i = \min(T_i, C_i)$ for $i = 1, \ldots, n$. The distribution of C_i does not depend on any of the unknown parameters of T_i. Parametric inference for such data are usually based on likelihood methods and their asymptotic theory. The censored log-likelihood $l(\boldsymbol{\theta})$ for the model parameters is

$$l(\boldsymbol{\theta}) = r\log(\alpha\beta) + \sum_{i\in F}\{(\gamma-1)\log(t_i) + \log(\gamma+\lambda t_i) + \lambda t_i - \alpha t_i^{\gamma}\exp(\lambda t_i)\}$$
$$+ (\beta-1)\sum_{i\in F}\log\left[1-\exp\{-\alpha t_i^{\gamma}\exp(\lambda t_i)\}\right] + \sum_{i\in C}\log\left[1-\left\{1-\exp(-\alpha t_i^{\gamma}\exp(\lambda t_i))\right\}^{\beta}\right],$$

where r is the number of failures and F and C denote the uncensored and censored sets of observations, respectively.

The score functions for the parameters α, γ, λ and β are given by

$$U_\alpha(\boldsymbol{\theta}) = r\alpha^{-1} + \sum_{i\in F}(\dot{v}_i)_\alpha\{v_i^{-1} - (\beta-1)(1-v_i)^{-1}\} + \sum_{i\in C}\beta(1-v_i)^{\beta-1}\{1-(1-v_i)^{\beta}\}^{-1}(\dot{v}_i)_\alpha,$$
$$U_\gamma(\boldsymbol{\theta}) = \sum_{i\in F}\{\log(t_i) + (\gamma+\lambda t_i)^{-1}\} + \sum_{i\in F}(\dot{v}_i)_\gamma\{v_i^{-1} - (\beta-1)(1-v_i)^{-1}\}$$
$$+ \sum_{i\in C}\beta(1-v_i)^{\beta-1}\{1-(1-v_i)^{\beta}\}^{-1}(\dot{v}_i)_\gamma,$$
$$U_\lambda(\boldsymbol{\theta}) = \sum_{i\in F}t_i\{1 + (\gamma+\lambda t_i)^{-1}\} + \sum_{i\in F}(\dot{v}_i)_\lambda\{v_i^{-1} - (\beta-1)(1-v_i)^{-1}\}$$
$$+ \sum_{i\in C}\beta(1-v_i)^{\beta-1}\{1-(1-v_i)^{\beta}\}^{-1}(\dot{v}_i)_\lambda,$$
$$U_\beta(\boldsymbol{\theta}) = r\beta^{-1} + \sum_{i\in F}\log(1-v_i) - \sum_{i\in C}\log(1-v_i)(1-v_i)^{\beta}\{1-(1-u_i)^{\beta}\}^{-1},$$

where

$$v_i = \exp\{-\alpha t_i^{\gamma}\exp(\lambda t_i)\}, \qquad (\dot{v}_i)_\alpha = -v_i t_i^{\gamma}\exp(\lambda t_i), \qquad (\dot{v}_i)_\gamma = -v_i\alpha t_i^{\gamma}\log(t_i)\exp(\lambda t_i)$$
$$\text{and} \quad (\dot{v}_i)_\lambda = -v_i\alpha t_i^{\gamma+1}\exp(\lambda t_i).$$

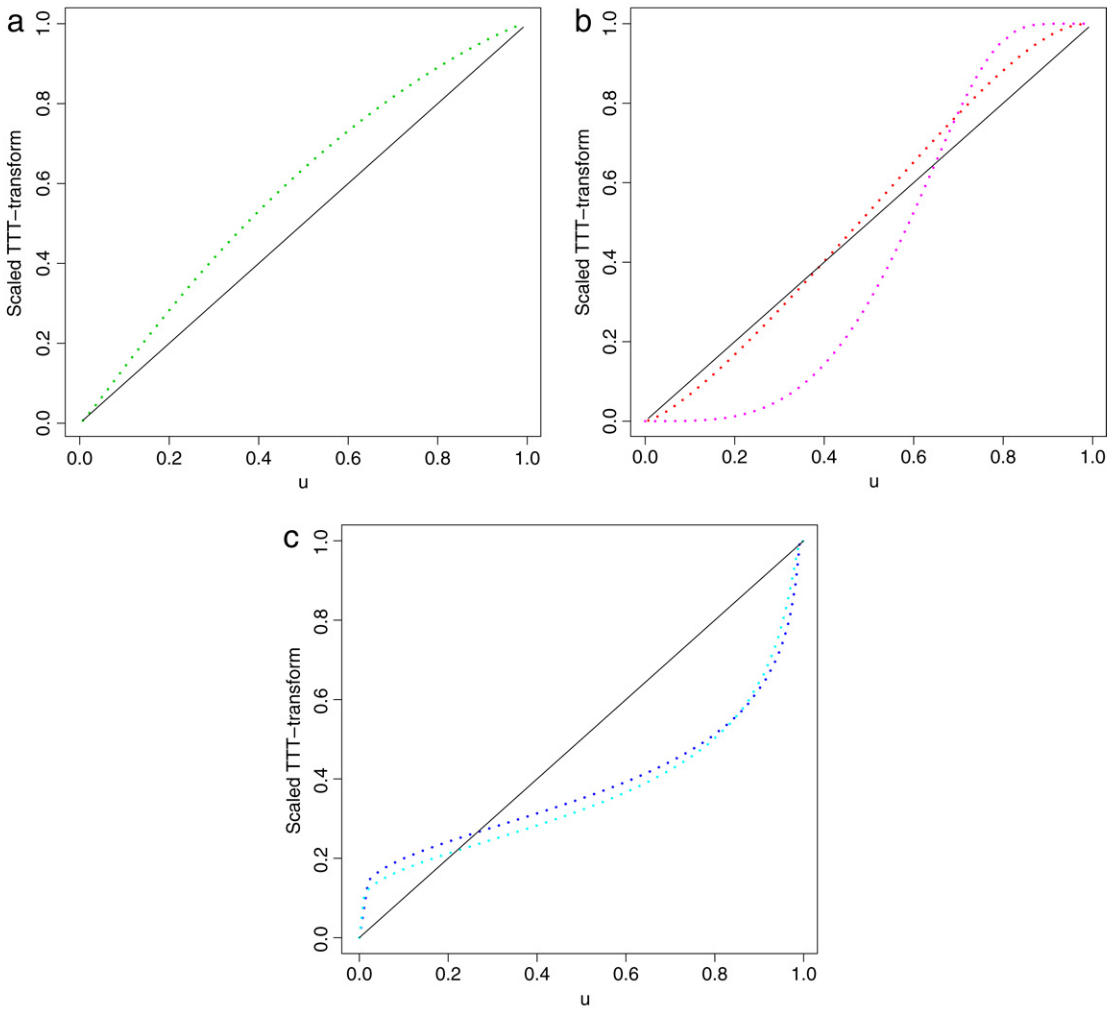

Fig. 3. Different forms of $\phi_F(u)$ for the GMW distribution. (a) Increasing shape ($\alpha = 0.5$, $\gamma = 1.5$, $\lambda = 0.1$, $\beta = 0.5$). (b) Bathtub shape ($\alpha = 0.5$, $\gamma = 0.8$, $\lambda = 0.1$, $\beta = 0.8$). (c) Unimodal shape ($\alpha = 10$, $\gamma = 0.48$, $\lambda = 0.1$, $\beta = 15$).

The maximum likelihood estimate (MLE) $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is obtained by solving the nonlinear likelihood equations $U_\alpha(\boldsymbol{\theta}) = 0$, $U_\gamma(\boldsymbol{\theta}) = 0$, $U_\lambda(\boldsymbol{\theta}) = 0$ and $U_\beta(\boldsymbol{\theta}) = 0$. These equations cannot be solved analytically and statistical software can be used to solve the equations numerically. We can use iterative techniques such as a Newton–Raphson type algorithm to obtain the estimate $\widehat{\boldsymbol{\theta}}$. We employed the programming matrix language Ox (Doornik, 2007).

For interval estimation of $(\alpha, \gamma, \lambda, \beta)$ and tests of hypotheses on these parameters we obtain the observed information matrix since the expected information matrix is very complicated and will require numerical integration. The 4×4 observed information matrix $J(\boldsymbol{\theta})$ is

$$J(\boldsymbol{\theta}) = -\begin{pmatrix} \mathbf{L}_{\alpha\alpha} & \mathbf{L}_{\alpha\gamma} & \mathbf{L}_{\alpha\lambda} & \mathbf{L}_{\alpha\beta} \\ . & \mathbf{L}_{\gamma\gamma} & \mathbf{L}_{\gamma\lambda} & \mathbf{L}_{\gamma\beta} \\ . & . & \mathbf{L}_{\lambda\lambda} & \mathbf{L}_{\lambda\beta} \\ . & . & . & \mathbf{L}_{\beta\beta} \end{pmatrix},$$

where the elements of this matrix are given in the Appendix.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of

$$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \text{ is } N_4(\mathbf{0}, I(\boldsymbol{\theta})^{-1}),$$

where $I(\boldsymbol{\theta})$ is the expected information matrix. This asymptotic behavior is valid if $I(\boldsymbol{\theta})$ is replaced by $J(\widehat{\boldsymbol{\theta}})$, i.e., the observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$. The asymptotic multivariate normal $N_4(\mathbf{0}, J(\widehat{\boldsymbol{\theta}})^{-1})$ distribution can be used to construct approximate confidence intervals and confidence regions for the individual parameters and for the hazard rate and survival functions. The asymptotic normality is also useful for testing goodness-of-fit of the GMW distribution and for comparing

Table 1
MLEs of the model parameters for the serum-reversal data, the corresponding SE (given in parentheses) and the measures AIC, BIC and CAIC

Model	α	γ	λ	β	AIC	BIC	CAIC
Generalized Modified Weibull	7.4e−06 (1.5e−07)	0.649 (0.471)	0.023 (0.006)	0.491 (0.116)	779.8	795.7	795.8
Modified Weibull	0.002 (0.000)	0.356 (0.297)	0.014 (0.002)	1	781.4	793.3	793.4
Exponentiated Weibull	1.2e−09 (3.3e−10)	3.557 (0.477)	0	0.785 (0.138)	808.2	820.2	820.2
Exponentiated Exponential	0.007 (6.0e−05)	1	0	3.701 (0.638)	839.3	847.3	847.3
Weibull	1.8e−08 (3.3e−98)	3.113 (0.325)	0	1	808.0	816.0	816.0
Generalized Rayleigh	1.3e−05 (2.0e−07)	0	2	1.455 (0.215)	817.3	825.4	825.3

this distribution with some of its special sub-models using one of the three well-known asymptotically equivalent test statistics—namely, the likelihood ratio (LR) statistic, Wald and Rao score statistics.

We can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain the LR statistics for testing some sub-models of the GMW distribution. For example, we may use the LR statistic to check if the fit using the GMW distribution is statistically "superior" to a fit using the MW or EW distribution for a given data set. In any case, hypothesis testing of the type $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ *versus* $H : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$, can be performed by using any of the above three asymptotically equivalent test statistics. For example, the test of $H_0 : \beta = 1$ *versus* $H : \beta \neq 1$ is equivalent to compare the MW distribution with the GMW distribution. In this case, the LR statistic is

$$w = 2\{l(\widehat{\alpha}, \widehat{\gamma}, \widehat{\lambda}, \widehat{\beta}) - l(\widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\lambda}, 1)\},$$

where $\widehat{\alpha}$, $\widehat{\gamma}$, $\widehat{\lambda}$ and $\widehat{\beta}$ are the MLEs under H and $\widetilde{\alpha}$, $\widetilde{\gamma}$ and $\widetilde{\lambda}$ are the estimates under H_0. Mudholkar et al. (1995) in their discussion of the classical bus-motor-failure data, noted the curious aspect in which the larger EW distribution provides an inferior fit as compared to the smaller Weibull distribution.

8. Two application examples

8.1. Serum-reversal data

The data set refers to the serum-reversal time (days) of 148 children contaminated with HIV from vertical transmission at the university hospital of the Ribeirão Preto School of Medicine (Hospital das Clínicas da Faculdade de Medicina de Ribeirão Preto) from 1986 to 2001 (Silva, 2004; Perdoná, 2006). Serum-reversal is a process of disappearance of anti-HIV antibodies (antitoxin) in blood (neutralization of anti-HIV serology) in an individual who previously showed positive anti-HIV serology. Serum-reversal can occur in children born from mothers infected with HIV. Their children are born with positive anti-HIV serology (vertical HIV transmission), which can occur due to the intrauterine or intra-parturition transplacental transmission of the mother's antibodies to her baby during labor or in the period following childbirth while the infant is breastfed. After a few months, the mother's antibodies are eliminated and the anti-HIV serology changes from positive to negative. In fact, the child had never been infected with HIV. Fig. 4a shows that the TTT-plot for the data set has first a convex shape and then a concave shape. It indicates a bathtub-shaped hazard rate function. Hence, the GMW distribution could be an appropriate model for the fitting of such data. Table 1 gives the MLEs (and the corresponding standard errors in parentheses) of the parameters and the values of the following statistics for some models: AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and CAIC (Consistent Akaike Information Criterion). The computations were done using the subroutine MAXBFGS in Ox. From the values of these statistics, we may infer that the top two models are the GMW and MW distributions and the other distributions are far worse for this data set.

In order to assess if the model is appropriate, we plot in Fig. 4b the empirical survival function and the estimated survival function of the GMW distribution which provides a good fit for the data under analysis. Additionally, the estimated hazard rate function in Fig. 4c is a bathtub-shaped curve.

8.2. Radiotherapy data

The data set refers to the survival time (days) for cancer patients ($n = 51$) undergoing radio therapy (Louzada-Neto et al., 2001). Fig. 5a shows that the TTT-plot for the data set has first a concave shape and then a convex shape, which indicates that the hazard rate function has a unimodal shape. Hence, an appropriate model for fitting such data could be the GMW distribution. Table 2 gives the MLEs (and the corresponding standard errors in parentheses) of the parameters and the previous statistics for various models. The preferred models based on the AIC and CAIC criterions are the EW and GMW distributions. However, the lower values of BIC for the EW and MW distributions indicate that these models could be chosen

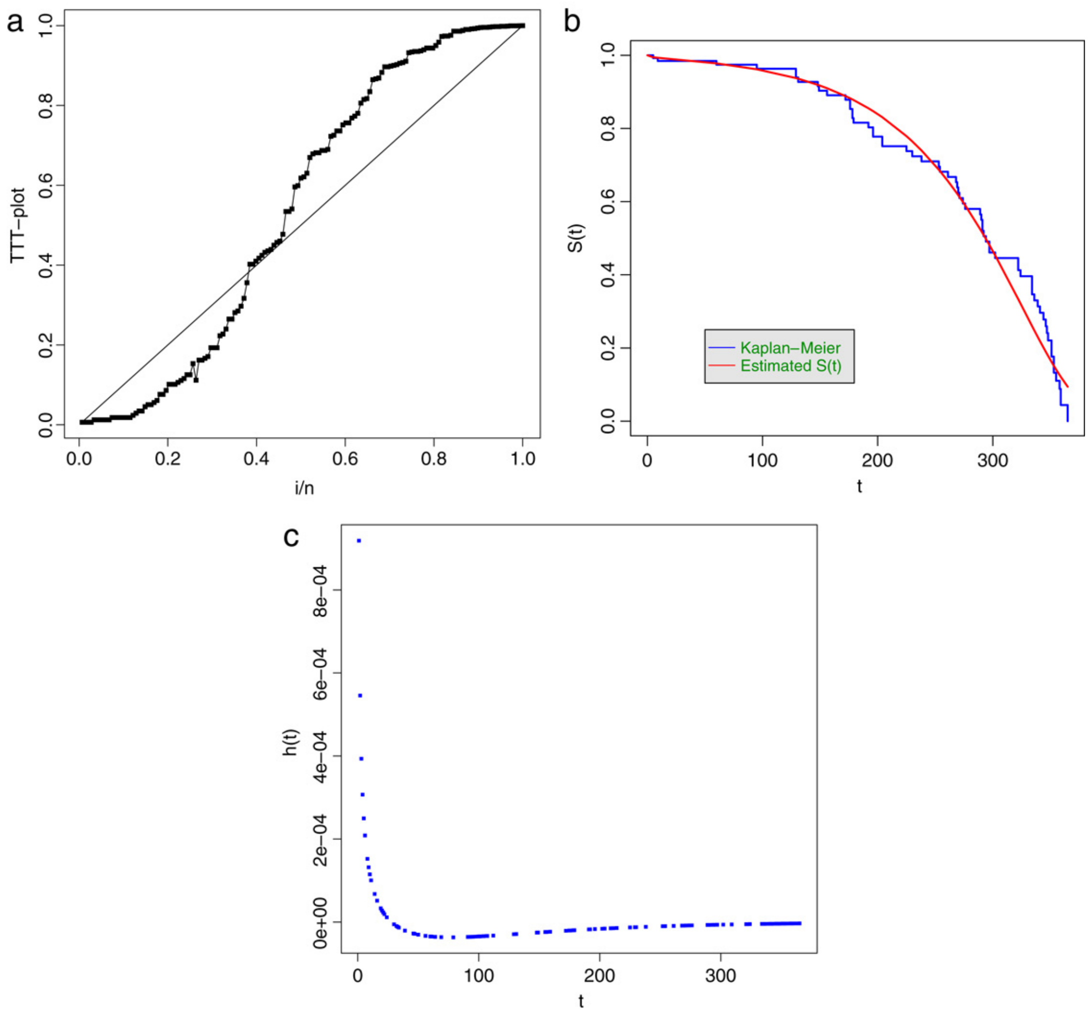

Fig. 4. (a) TTT-plot on serum-reversal data. (b) Estimated survival function and the empirical survival for serum-reversal data. (c) Estimated hazard rate function for the serum-reversal data.

Table 2
MLEs of the model parameters for the radiotherapy data, the corresponding SE (given in parentheses) and the measures AIC, BIC and CAIC

Model	α	γ	λ	β	AIC	BIC	CAIC
Generalized Modified Weibull	0.168 (0.049)	0.483 (0.297)	0.0002 (0.0001)	6.326 (2.662)	593.3	601.0	594.1
Modified Weibull	0.001 (0.0001)	1.245 (0.181)	0.001 (0.0002)	1	594.4	600.1	594.9
Exponentiated Weibull	0.656 (0.109)	0.293 (0.140)	0	18.359 (8.366)	592.1	597.9	592.6
Exponentiated Exponential	0.002 (0.0007)	1	0	1.018 (0.018)	598.2	602.1	598.5
Weibull	0.004 (0.0003)	0.930 (0.110)	0	1	597.8	601.7	598.1
Generalized Rayleigh	0.000001 (5.0e−08)	0	2	0.342 (0.057)	607.4	611.3	607.7

as the best models to fit the data. In any case, since the differences of the three statistics are quite small for the GMW, MW e EW distributions, the proposed distribution seems to be a very competitive model for lifetime data analysis. In Fig. 5b, we plot the empirical survival function and the estimated survival function for the GMW distribution which gives a satisfactory fit. Furthermore, Fig. 5c shows that the estimated hazard rate function has a unimodal shape.

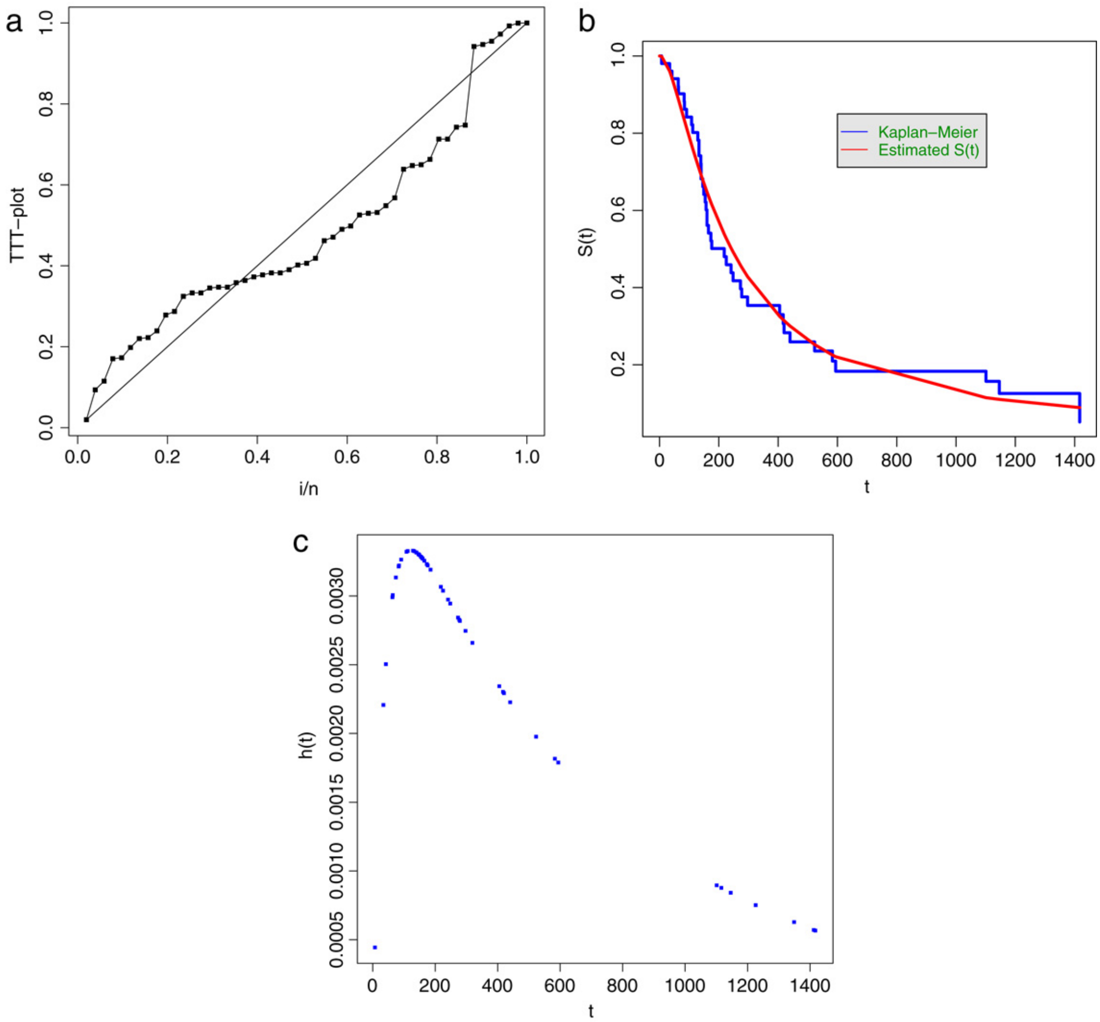

Fig. 5. (a) TTT-plot on the radiotherapy data. (b) Estimated survival function from the fitting of the GMW distribution and the empirical survival function for the radiotherapy data. (c) Estimated hazard rate function for the radiotherapy data.

9. Conclusions

We introduce a four parameter lifetime distribution called "a generalized modified Weibull (GMW) distribution" which is a simple extension of the modified Weibull distribution (Lai et al., 2003). The new model extends several distributions widely used in the lifetime literature and is more flexible than the exponentiated Weibull and the modified Weibull sub-models. The proposed distribution could have increasing, decreasing, bathtub and unimodal hazard rate functions. It is then useful to model lifetime with a bathtub-shaped hazard rate function. We provide a mathematical treatment of this distribution including the order statistics. Explicit algebraic formulae for the moments which hold in generality for any parameter values are given. We provide finite weighted sums for the moments of the order statistics. The application of the proposed distribution is straightforward. We discuss maximum likelihood estimation and testing of hypotheses for the model parameters. The GMW distribution permits testing the goodness-of-fit of some well-known distributions in reliability analysis by taking these distributions as sub-models. The practical relevance and applicability of the new model are demonstrated in two applications and whether some special sub-models could provide similar fit while being more parsimonious models. These applications demonstrate the usefulness of the GMW distribution, and with the use of modern computer resources with analytic and numerical capabilities, it can be an adequate tool comprising the arsenal of distributions for lifetime analysis.

Acknowledgments

The authors are grateful to the editor and two referees for their helpful comments and suggestions.

Appendix

The elements of the observed information matrix $J(\boldsymbol{\theta})$ for the parameters $(\alpha, \gamma, \lambda, \beta)$ are

$$\mathbf{L}_{\alpha\alpha} = -r\alpha^{-2} + \sum_{i\in F}\left[-\left\{\frac{(\dot{v}_i)_\alpha}{v_i}\right\}^2 + \frac{(\ddot{v}_i)_{\alpha\alpha}}{v_i}\right] - (\beta-1)\sum_{i\in F}\left[\left\{\frac{(\dot{v}_i)_\alpha}{1-v_i}\right\}^2 + \frac{(\ddot{v}_i)_{\alpha\alpha}}{1-v_i}\right]$$
$$+\beta\sum_{i\in C}\left[\frac{-(1-v_i)^{\beta-2}\left\{\beta-1+(1-v_i)^\beta\right\}\{(\dot{v}_i)_\alpha\}^2}{\left\{1-(1-v_i)^\beta\right\}^2} + \frac{(1-v_i)^{\beta-1}(\ddot{v}_i)_{\alpha\alpha}}{\left\{1-(1-v_i)^\beta\right\}}\right],$$

$$\mathbf{L}_{\alpha\gamma} = \sum_{i\in F}\left\{-\frac{(\dot{v}_i)_\alpha(\dot{v}_i)_\gamma}{v_i^2} + \frac{(\ddot{v}_i)_{\alpha\gamma}}{v_i}\right\} - (\beta-1)\sum_{i\in F}\left\{\frac{(\dot{v}_i)_\alpha(\dot{v}_i)_\gamma}{(1-v_i)^2} + \frac{(\ddot{v}_i)_{\alpha\gamma}}{1-v_i}\right\}$$
$$+\beta\sum_{i\in C}\left[\frac{-(1-v_i)^{\beta-2}\left\{\beta-1+(1-v_i)^\beta\right\}(\dot{v}_i)_\alpha(\dot{v}_i)_\gamma}{\left\{1-(1-v_i)^\beta\right\}^2} + \frac{(1-v_i)^{\beta-1}(\ddot{v}_i)_{\alpha\gamma}}{\left\{1-(1-v_i)^\beta\right\}}\right],$$

$$\mathbf{L}_{\alpha\lambda} = \sum_{i\in F}\left\{-\frac{(\dot{v}_i)_\alpha(\dot{v}_i)_\lambda}{v_i^2} + \frac{(\ddot{v}_i)_{\alpha\lambda}}{v_i}\right\} - (\beta-1)\sum_{i\in F}\left\{\frac{(\dot{v}_i)_\alpha(\dot{v}_i)_\lambda}{(1-v_i)^2} + \frac{(\ddot{v}_i)_{\alpha\lambda}}{1-v_i}\right\}$$
$$+\beta\sum_{i\in C}\left[\frac{-(1-v_i)^{\beta-2}\left\{\beta-1+(1-v_i)^\beta\right\}(\dot{v}_i)_\alpha(\dot{v}_i)_\lambda}{\left\{1-(1-v_i)^\beta\right\}^2} + \frac{(1-v_i)^{\beta-1}(\ddot{v}_i)_{\alpha\lambda}}{\left\{1-(1-v_i)^\beta\right\}}\right],$$

$$\mathbf{L}_{\alpha\beta} = \sum_{i\in F}\frac{(\dot{v}_i)_\alpha}{(1-v_i)} + \sum_{i\in C}\frac{(\dot{v}_i)_\alpha(1-v_i)^{\beta-1}\left\{1+\beta\log(1-v_i)-(1-v_i)^\beta\right\}}{\left\{1-(1-v_i)^\beta\right\}^2},$$

$$\mathbf{L}_{\gamma\gamma} = -\sum_{i\in F}(\gamma+\lambda t_i)^{-2} + \sum_{i\in F}\left[-\left\{\frac{(\dot{v}_i)_\gamma}{v_i}\right\}^2 + \frac{(\ddot{v}_i)_{\gamma\gamma}}{v_i}\right] - (\beta-1)\sum_{i\in F}\left[\left\{\frac{(\dot{v}_i)_\gamma}{1-v_i}\right\}^2 + \frac{(\ddot{v}_i)_{\gamma\gamma}}{1-v_i}\right]$$
$$+\beta\sum_{i\in C}\left[\frac{-(1-v_i)^{\beta-2}\left\{\beta-1+(1-v_i)^\beta\right\}\left\{(\dot{v}_i)_\gamma\right\}^2}{\left\{1-(1-v_i)^\beta\right\}^2} + \frac{(1-v_i)^{\beta-1}(\ddot{v}_i)_{\gamma\gamma}}{\left\{1-(1-v_i)^\beta\right\}}\right],$$

$$\mathbf{L}_{\lambda\gamma} = -\sum_{i\in F}(\gamma+\lambda t_i)^{-2}t_i + \sum_{i\in F}\left\{-\frac{(\dot{v}_i)_\lambda(\dot{v}_i)_\gamma}{v_i^2} + \frac{(\ddot{v}_i)_{\lambda\gamma}}{v_i}\right\} - (\beta-1)\sum_{i\in F}\left\{\frac{(\dot{v}_i)_\lambda(\dot{v}_i)_\gamma}{(1-v_i)^2} + \frac{(\ddot{v}_i)_{\lambda\gamma}}{1-v_i}\right\}$$
$$+\beta\sum_{i\in C}\left[\frac{-(1-v_i)^{\beta-2}\left\{\beta-1+(1-v_i)^\beta\right\}(\dot{v}_i)_\lambda(\dot{v}_i)_\gamma}{\left\{1-(1-v_i)^\beta\right\}^2} + \frac{(1-v_i)^{\beta-1}(\ddot{v}_i)_{\lambda\gamma}}{\left\{1-(1-v_i)^\beta\right\}}\right],$$

$$\mathbf{L}_{\gamma\beta} = -\sum_{i\in F}\frac{(\dot{v}_i)_\gamma}{(1-v_i)} + \sum_{i\in C}\frac{(\dot{v}_i)_\gamma(1-v_i)^{\beta-1}\left\{1+\beta\log(1-v_i)-(1-v_i)^\beta\right\}}{\left\{1-(1-v_i)^\beta\right\}^2},$$

$$\mathbf{L}_{\lambda\lambda} = -\sum_{i\in F}(\gamma+\lambda t_i)^{-2}t_i^2 + \sum_{i\in F}\left[-\left\{\frac{(\dot{v}_i)_\lambda}{v_i}\right\}^2 + \frac{(\ddot{v}_i)_{\lambda\lambda}}{v_i}\right] - (\beta-1)\sum_{i\in F}\left[\left\{\frac{(\dot{v}_i)_\lambda}{1-v_i}\right\}^2 + \frac{(\ddot{v}_i)_{\lambda\lambda}}{1-v_i}\right]$$
$$+\beta\sum_{i\in C}\left[\frac{-(1-v_i)^{\beta-2}\left\{\beta-1+(1-v_i)^\beta\right\}\{(\dot{v}_i)_\lambda\}^2}{\left\{1-(1-v_i)^\beta\right\}^2} + \frac{(1-v_i)^{\beta-1}(\ddot{v}_i)_{\lambda\lambda}}{\left\{1-(1-v_i)^\beta\right\}^2}\right],$$

$$\mathbf{L}_{\lambda\beta} = -\sum_{i\in F}\frac{(\dot{v}_i)_\lambda}{(1-v_i)} + \sum_{i\in C}\frac{(\dot{v}_i)_\lambda(1-v_i)^{\beta-1}\left\{1+\beta\log(1-v_i)-(1-v_i)^\beta\right\}}{\left\{1-(1-v_i)^\beta\right\}^2},$$

$$\mathbf{L}_{\beta\beta} = -r\beta^{-2} - \sum_{i\in C}\frac{(1-v_i)^\beta\{\log(1-v_i)\}^2}{\left\{1-(1-v_i)^\beta\right\}},$$

where

$$v_i = \exp\{-\alpha t_i^\gamma\exp(\lambda t_i)\}, \qquad (\dot{v}_i)_\alpha = -v_i t_i^\gamma\exp(\lambda t_i),$$
$$(\dot{v}_i)_\gamma = -v_i\alpha t_i^\gamma\log(t_i)\exp(\lambda t_i), \qquad (\dot{v}_i)_\lambda = -v_i\alpha t_i^{\gamma+1}\exp(\lambda t_i),$$
$$(\ddot{v}_i)_{\alpha\alpha} = -t_i^\gamma\exp(\lambda t_i)(\dot{v}_i)_\alpha, \qquad (\ddot{v}_i)_{\gamma\gamma} = -\alpha t_i^\gamma\log(t_i)\exp(\lambda t_i)\{\log(t_i)u_i + (\dot{v}_i)_\gamma\},$$
$$(\ddot{v}_i)_{\lambda\lambda} = -\alpha t_i^{\gamma+1}\exp(\lambda t_i)\{t_i u_i + (\dot{v}_i)_\lambda\}, \qquad (\ddot{v}_i)_{\gamma\alpha} = -t_i^\gamma\exp(\lambda t_i)\{(\dot{v}_i)_\gamma + u_i\log(t_i)\},$$
$$(\ddot{v}_i)_{\lambda\alpha} = -t_i^\gamma\exp(\lambda t_i)\{(\dot{v}_i)_\lambda + u_i t_i\} \quad \text{and} \quad (\ddot{v}_i)_{\gamma\lambda} = -\alpha t_i^\gamma\log(t_i)\exp(\lambda t_i)\{(\dot{v}_i)_\lambda + u_i t_i\}.$$

References

Aarset, M.V., 1987. How to identify bathtub hazard rate. IEEE Transactions on Reliability 36, 106–108.

Barlow, R.E., Campo, R., 1975. Total time on test processes and applications to failure data analysis. In: Reliability and Fault Tree Analysis. Society for Industrial and Applied Mathematics, pp. 451–481.

Bebbington, M., Lai, C.D., Zitikis, R., 2007. A flexible Weibull extension. Reliability Engineering and System Safety 92, 719–726.

Choudhury, A., 2005. A simple derivation of moments of the exponentiated Weibull distribution. Metrika 62, 17–22.

Doornik, J., 2007. Ox: An Object-Oriented Matrix Programming Language. International Thomson Business Press.

Glaser, R.E., 1980. Bathtub and related failure rate characterizations. Journal of the American Statistical Association 75, 667–672.

Gupta, R.D., Kundu, D., 1999. Generalized exponential distributions. Australian and New Zealand Journal of Statistics 41, 173–188.

Gupta, R.D., Kundu, D., 2001. Exponentiated exponential distribution: An alternative to gamma and Weibull distributions. Biometrical Journal 43, 117–130.

Haupt, E., Schabe, H., 1992. A new model for a lifetime distribution with bathtub shaped failure rate. Microelectronics and Reliability 32, 633–639.

Hjorth, U., 1980. A reliability distribution with increasing, decreasing, constant and bathtub failure rates. Technometrics 22, 99–107.

Kundu, D., Rakab, M.Z., 2005. Generalized Rayleigh distribution: Different methods of estimation. Computational Statistics and Data Analysis 49, 187–200.

Lai, C.D., Moore, T., Xie, M., 1998. The beta integrated model. In: Proceedings International Workshop on Reliability Modeling and Analysis–From Theory to Practice, pp. 153–159.

Lai, C.D., Xie, M., Murthy, D.N.P., 2003. A modified Weibull distribution. IEEE Transactions on Reliability 52, 33–37.

Louzada-Neto, F., Mazucheli, J., Achcar, J.A., 2001. Uma Introdução à Análise de Sobrevivência e Confiabilidade. XXVIII Jornadas Nacionales de Estadística, Chile.

Mudholkar, G.S., Srivastava, D.K., Friemer, M., 1995. The exponentiated Weibull family: A reanalysis of the bus-motor-failure data. Technometrics 37, 436–445.

Mudholkar, G.S., Srivastava, D.K., Kollia, G.D., 1996. A generalization of the Weibull distribution with application to the analysis of survival data. Journal of the American Statistical Association 91, 1575–1583.

Nadarajah, S., 2005. On the moments of the modified Weibull distribution. Reliability Engineering and System Safety 90, 114–117.

Nadarajah, S., Kotz, S., 2005. On some recent modifications of Weibull distribution. IEEE Transactions Reliability 54, 561–562.

Nelson, W., 1982. Lifetime Data Analysis. Wiley, New York.

Perdoná, G.S.C., 2006. Modelos de Riscos Aplicados à Análise de Sobrevivência. Doctoral Thesis, Institute of Computer Science and Mathematics, University of São Paulo, Brasil (in Portuguese).

Pham, H., Lai, C.D., 2007. On recent generalizations of the Weibull distribution. IEEE Transactions on Reliability 56, 454–458.

Rajarshi, S., Rajarshi, M.B., 1988. Bathtub distributions: A review. Communications in Statistics-Theory and Methods 17, 2521–2597.

Silva, A.N.F., 2004. Estudo evolutivo das crianças expostas ao HIV e notificadas pelo núcleo de vigilância epidemiológica do HCFMRP-USP. M.Sc. Thesis. University of São Paulo, Brazil.

Xie, M., Lai, C.D., 1995. Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. Reliability Engineering and System Safety 52, 87–93.

Xie, M., Tang, Y., Goh, T.N., 2002. A modified Weibull extension with bathtub failure rate function. Reliability Engineering and System Safety 76, 279–285.